Invariant Distribution

- Finite State Space

$$\underline{\pi}' P = \pi'$$

$$\underline{\pi} = [\pi_i] \qquad P = (p_{ij})$$

$$\pi_i = P(X_0 = i)$$
$$p_{ij} = P(X_1 = j \mid X_0 = i)$$

$$P' \underline{\pi} = \underline{\pi}$$

Eigen vector with eigen value $= 1$.

- Countable State Space

$$\pi_j = \sum_{i=1}^{\infty} \pi_i \, p_{ij} \qquad j = 1, 2, \ldots$$

Sufficient condition
$\pi_i p_{ij} = \pi_j p_{ji} \quad \forall ij$

- Continuous State Space

$$\pi(x) = \int_S \pi(y)\, h(x \mid y)\, dy$$

$h(x \mid y) \leftarrow$ transition kernel

Simple condition for invariance

$$\boxed{\pi(y)\, h(x \mid y) = \pi(x)\, h(y \mid x) \quad \forall x, y \in S}$$

Proof

$$\int_S \pi(y)\, h(x \mid y)\, dy = \int_S \pi(x)\, h(y \mid x)\, dy = \pi(x) \int_S h(y \mid x)\, dy = \pi(x).$$

The Gibbs Sampler

Update Mechanism: well defined procedure that makes a random change in the current state of the system according to a probability distribution that depends only on the current state.

The resulting sequence will then be a Markov process.

Invariant Update mechanism: An update mechanism that preserves the distribution of the updated state.

Gibbs Sampler: let $\underline{X}_0$ be a random element from S

Consider a function $h(\underline{X}_0)$. For example

$$h(X_{01}, X_{02}, X_{03}) = (X_{02}, X_{03})$$

Define $\underline{X}_1$ as a new element of S generated from the distribution of $\underline{X}_0$ given $h(\underline{X}_0)$.

In the case of our example

$$\underline{X}_1 = \begin{pmatrix} X_{11} \\ X_{02} \\ X_{03} \end{pmatrix} \quad \text{where } X_{11} \sim \mathcal{L}(X_{01} \mid X_{02}, X_{03})$$

For instace suppose we wish to preserve the distribution $N\left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 0.5 & 0.4 \\ 0.5 & 1 & 0.5 \\ 0.4 & 0.5 & 1 \end{pmatrix}\right)$

and suppose that $\underline{X}_0 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$.

Then

$$X_{01} \mid X_{02}=0, X_{03}=1 \sim N(0.4, 0.72)$$

$$\text{Mean} = (0.4 \quad 0.2)\begin{pmatrix}0\\1\end{pmatrix} = \boxed{0.4}, \quad \text{Var} = 1 - (0.4 \quad 0.2)\begin{pmatrix}0.5\\0.4\end{pmatrix}$$

$$= 1 - 0.28$$

$$= \boxed{0.72}$$

$$\left(\begin{array}{c|cc} 1 & 0.5 & 0.4 \\ \hline 0.5 & 1 & 0.5 \\ 0.4 & 0.5 & 1 \end{array}\right)$$

$$\Sigma_{12} = (0.5 \quad 0.4)$$

$$\Sigma_{22} = \begin{pmatrix}1 & 0.5\\0.5 & 1\end{pmatrix}$$

$$\Sigma_{22}^{-1} = \frac{1}{1-0.25}\begin{pmatrix}1 & -0.5\\-0.5 & 1\end{pmatrix} = \frac{4}{3}\begin{pmatrix}1 & -0.5\\-0.5 & 1\end{pmatrix}$$

$$\Sigma_{12}\Sigma_{22}^{-1} = \frac{4}{3}(0.5 \quad 0.4)\begin{pmatrix}1 & -0.5\\-0.5 & 1\end{pmatrix} = \frac{4}{3}(0.3 \quad 0.15)$$

$$= (0.4, \; 0.20)$$

Suppose that $\underline{X}_0 \sim F$

Let $h(\underline{X}_0)$ be a function such that we can easily generate a new vector $\underline{X}_1$ from the conditional distribution of $\underline{X}_0 \mid h(\underline{X}_0)$.

Key observation: $\boxed{\underline{X}_1 \mid h(\underline{X}_0) =_d \underline{X}_0 \mid h(\underline{X}_0)}$ by definition of $\underline{X}_1$.

Let $g(\underline{X})$ be any measurable function.

We notice that

$$
\begin{aligned}
E[g(\underline{X}_1)] &= E\{E[g(\underline{X}_1) \mid h(\underline{X}_0)]\} \\
&= E\{E[g(\underline{X}_0) \mid h(X_0)]\} \\
&= E[g(X_0)]
\end{aligned}
$$

$$\Rightarrow \boxed{\underline{X}_1 =_d \underline{X}_0}$$

Conclusion:

Example: Bayesian Inference for $N(\mu, \lambda^{-1})$

Given (μ, λ^{-1}), the data $X_1, \dots, X_n$ are iid $N(\mu, \lambda^{-1})$.

The prior distribution is

$$\lambda \sim \text{Gamma}(\alpha, \beta)$$
$$\mu / \lambda \sim N(\gamma, \delta^{-1})$$

Recall: Gamma(α, β)

$$f(x) = \frac{\beta^{\alpha}}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}, \quad x > 0$$

Posterior Density

$$\pi(\mu,\lambda) = \lambda^{n/2} \exp\left\{-\frac{n\lambda s_n^2}{2} - \frac{n\lambda}{2}(\bar{x}-\mu)^2\right\}$$

$$\lambda^{\alpha-1} e^{-\beta\lambda} \exp\left\{-\frac{\delta}{2}(\mu-\gamma)^2\right\}$$

Can be shown that

$$\lambda \mid \mu \sim \text{Gamma}\left(\alpha + \frac{n}{2},\ \beta + \frac{n s_n^2}{2} + \frac{n}{2}(\bar{x}-\mu)^2\right)$$

$$\mu \mid \lambda \sim N\left(\frac{n\lambda\bar{x}+\delta\gamma}{n\lambda+\delta},\ \frac{1}{n\lambda+\delta}\right)$$