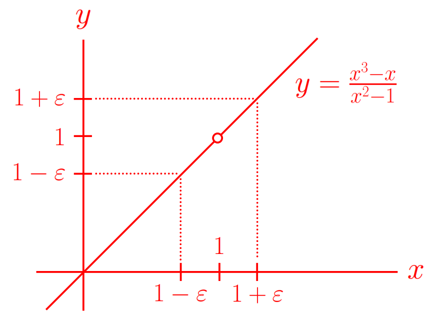
ASSIGNMENT 1

Solutions

1. Evaluate the limit $\lim_{x\to 1} \frac{x^3 - x}{x^2 - 1}$, and justify your answer using the definition of limit.

   Note that $\frac{x^3-x}{x^2-1} = x$ for $x \neq \pm 1$. We claim that $\lim_{x\to 1} \frac{x^3 - x}{x^2 - 1} = 1$.



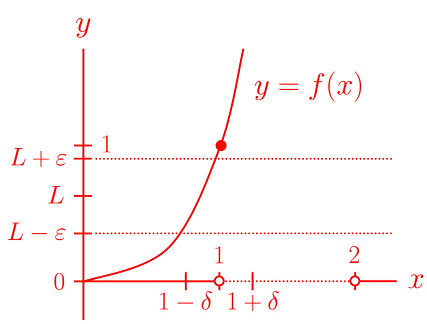
   Let $\varepsilon > 0$ be given. Can we guarantee $\frac{x^3-x}{x^2-1}$ to be within ε of 1 provided x is sufficiently close to 1? We can. Indeed, $\frac{x^3-x}{x^2-1}$ is guaranteed to be within ε of 1 provided x is within ε of 1.

2. Let $f(x) = \begin{cases} x^2 & \text{if } x \text{ has a decimal expansion containing the digit "1"} \\ 0 & \text{otherwise} \end{cases}$.

   (a) Prove using the definition of limit that $\lim_{x\to 0} f(x) = 0$.

   (b) Prove using the definition of limit that $\lim_{x\to 1} f(x)$ does not exist.

   Let $\varepsilon > 0$ be given. Can we guarantee $f(x)$ to be within ε of 0 provided x is sufficiently close to 0? We can. Indeed, $f(x)$ is guaranteed to be within ε of 0 provided x is within $\sqrt{\varepsilon}$ of 1; that is, provided $x \in (-\sqrt{\varepsilon}, \sqrt{\varepsilon})$. For if x has a decimal expansion containing the digit "1", then $f(x) = x^2 \in (0, \varepsilon)$; and if x does not have a decimal expansion containing the digit "1", then $f(x) = 0 \in (-\varepsilon, \varepsilon)$. This proves that $\lim_{x\to 0} f(x) = 0$.



To prove that $\lim_{x\to 1} f(x)$ does not exist, we examine the graph of the function around $x = 1$ more closely. To the left of $x = 1$, there are numbers that do not have a decimal expansion containing "1" — for example, $x = 0.9, 0.99, 0.999, \ldots$. In particular, there exist such numbers inside any interval $(1 - \delta, 1)$. On the other hand, in the interval $(1, 2)$ to the right of $x = 1$, *all* numbers have a decimal expansion containing "1".

Now suppose $\lim_{x\to 1} f(x) = L$. This means that, for any $\varepsilon > 0$, we can guarantee $f(x)$ to be within ε of L provided x is sufficiently close to 1.

But let $\varepsilon = \frac{1}{3}$, say. We cannot guarantee $f(x)$ to be within ε of 0, no matter how close x is to 1; for inside any interval $(1-\delta, 1+\delta)$, there exist numbers $x < 1$ that do not have a decimal expansion containing "1" (that is, such that $f(x) = 0$), and numbers $x > 1$ that have a decimal expansion containing "1" (that is, such that $f(x) = x^2 > 1$). Both sets of y-values cannot be contained in the "horizontal band" bounded by $y = L - \varepsilon$ and $y = L + \varepsilon$.

3. Periodically, you will be asked to post mathematical reflections on a blog. These reflections train your ability to explain abstract mathematical ideas, a key skill in higher-level mathematics.

   Instead of completing a third written question on this assignment, we want you to set up your blog using the UBC Blogs service and post a test message on it. You must do so before Friday, September 18 to receive full marks. There is a link to the UBC Blogs page on the course webpage.

   On your assignment submission, please include the URL of your blog.