ASSIGNMENT 10

Solutions

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1. Let $f(x) = xe^{-x^2}$.

   (a) Find the domain, intercepts, vertical and horizontal asymptotes, intervals of increase and decrease, and intervals of concavity of $f(x)$.

   (b) Sketch the graph of $f(x)$.

$f(x)$ has domain $(-\infty, \infty)$, and a single intercept $(0,0)$. Because it is continuous on its domain, it has no vertical asymptotes; however, it has a horizontal asymptote $y = 0$, which we determine using L'Hospital's Rule:

$$\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} \frac{x}{e^{x^2}} = \lim_{x\to\pm\infty} \frac{1}{2}xe^{x^2} = 0.$$

We observe that the derivative

$$f'(x) = e^{-x^2} - 2x^2e^{-x^2} = e^{-x^2}\left(1 - 2x^2\right)$$

vanishes when $x = \pm\frac{1}{\sqrt{2}}$. We determine the sign of $f'(x)$ between critical points.

x	$\left(-\infty, -\frac{1}{\sqrt{2}}\right)$	$\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$	$\left(\frac{1}{\sqrt{2}}, \infty\right)$
$f'(x)$	$-$	$+$	$-$
$f(x)$	decreasing	increasing	decreasing

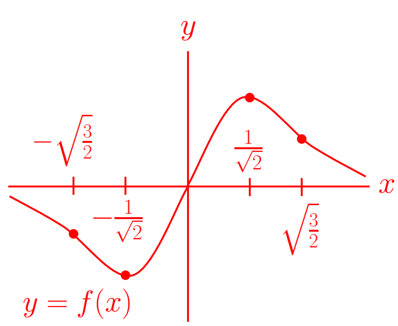
Finally, we observe that the second derivative

$$f''(x) = -2xe^{-x^2}\left(1 - 2x^2\right) - 4xe^{-x^2} = 2xe^{-x^2}\left(2x^2 - 3\right)$$

vanishes when $x = 0, \pm\sqrt{\frac{3}{2}}$. We determine the sign of $f''(x)$ between potential inflection points.

x	$\left(-\infty, -\sqrt{\frac{3}{2}}\right)$	$\left(-\sqrt{\frac{3}{2}}, 0\right)$	$\left(0, \sqrt{\frac{3}{2}}\right)$	$\left(\sqrt{\frac{3}{2}}, \infty\right)$
$f''(x)$	$-$	$+$	$-$	$+$
$f(x)$	concave down	concave up	concave down	concave up

We are now ready to sketch the graph, which we give below.



2. The curve E, described by the equation

$$y^2 = x^3 - 2x,$$

is an example of an *elliptic curve*, an object with deep connections to prime numbers.

(a) Find the domain, intercepts, vertical and horizontal asymptotes, intervals of increase and decrease, and intervals of concavity of $f(x) = \sqrt{x^3 - 2x}$.

(b) A curve is *nonsingular* if a single tangent line may be drawn at any point on the curve. (For example, $x^2 + y^2 = 1$ is nonsingular, but $y^2 = x^3$ is not — in the latter case, a single tangent line may not be drawn at $(0, 0)$.) Prove that E is nonsingular.

(c) Sketch the graph of E.

Note that $f(x) = \sqrt{x\left(x - \sqrt{2}\right)\left(x + \sqrt{2}\right)}$. Thus $f(x)$ has domain $\left[-\sqrt{2}, 0\right] \cup \left[\sqrt{2}, \infty\right)$, and intercepts $\left(-\sqrt{2}, 0\right)$, $(0, 0)$ and $\left(\sqrt{2}, 0\right)$. It is continuous on its domain, and has no vertical asymptotes. Since $\lim_{x\to\infty} f(x) = \infty$, it has no horizontal asymptotes.

We observe that the derivative

$$f'(x) = \frac{3x^2 - 2}{2x\sqrt{(x^2 - 2)}}$$

vanishes when $x = \pm\sqrt{\frac{2}{3}}$, and is undefined when $x = 0$ or $\pm\sqrt{2}$. However, only $x = -\sqrt{\frac{2}{3}}$ is in the interior of the domain. We have the following.

x	$\left(-\sqrt{2}, -\sqrt{\frac{2}{3}}\right)$	$\left(-\sqrt{\frac{2}{3}}, 0\right)$	$\left(\sqrt{2}, \infty\right)$
$f'(x)$	$+$	$-$	$+$
$f(x)$	increasing	decreasing	increasing

We observe that the second derivative

$$f''(x) = \frac{3x^4 - 12x^2 - 4}{4x\left((x^2 - 2)\right)^{3/2}}$$

is undefined when $x = 0$ or $\pm\sqrt{2}$. However, none of these points are in the interior of the domain. To see when the second derivative vanishes, we set $X = x^2$ and solve $3X^2 - 12X - 4 = 0$, getting $X = 2 \pm \frac{4}{\sqrt{3}}$. We consider only the positive value of X; and then, only the positive value $x = \sqrt{2 + \frac{4}{\sqrt{3}}}$, since all other solutions are imaginary or lie outside the domain. We have the following.

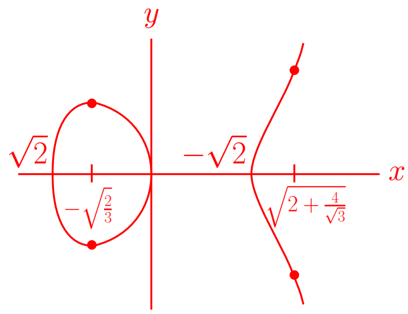
x	$\left(-\sqrt{2}, 0\right)$	$\left(\sqrt{2}, \sqrt{2 + \frac{4}{\sqrt{3}}}\right)$	$\left(\sqrt{2 + \frac{4}{\sqrt{3}}}, \infty\right)$
$f''(x)$	$-$	$-$	$+$
$f(x)$	concave down	concave down	concave up

Now E is simply $f(x)$ and $-f(x)$, "glued together". $f(x)$ — and therefore $-f(x)$ — is clearly nonsingular wherever its derivative is defined; that is, in the interior of the domain. It remains only to show that E is nonsingular when $x = 0$ or $\pm\sqrt{2}$. We claim that E has horizontal tangent lines at those three points. To see this, we differentiate E implicitly and with respect to y, getting

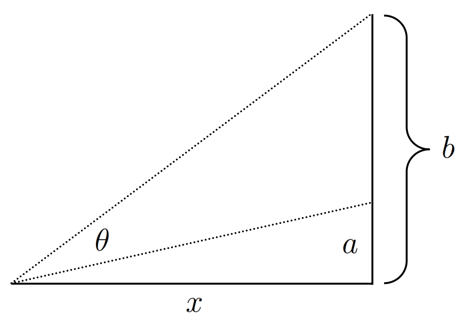
$$2y = 3x^2\frac{dx}{dy} - 2\frac{dx}{dy} = \frac{dx}{dy}\left(3x^2 - 2\right);$$

that is, $\frac{dx}{dy} = \frac{2y}{3x^2-2}$, which vanishes precisely when $y = \pm f(x)$ vanishes; that is, when $x = 0$ or $\pm\sqrt{2}$. (Incidentally, $\frac{dx}{dy}$ is undefined precisely when the tangent lines are horizontal.)

It remains only to sketch a graph of the curve.



3. In the 15[th] century, the mathematician Johannes Müller posed a version of this problem: given a picture of height $b-a$ hung a height a above eye level, how far should the viewer stand from the wall to get the best view of the picture? That is, in the figure below, maximize θ with respect to x.



In the given picture, it will be useful to define α as the angle subtended by the length a, and β as the angle subtended by the length b; thus $\theta = \beta - \alpha$. We wish to maximize θ with respect to x; or what is equivalent, since $\theta \in \left(0, \frac{\pi}{2}\right)$, we wish to maximize $\tan(\theta)$. Now

$$\tan(\beta - \alpha) = \frac{\tan(\beta) - \tan(\alpha)}{1 + \tan(\beta)\tan(\alpha)} = (b-a)\frac{x}{x^2+ab}.$$

(We may arrive at the first equality using the angle-sum formulas proven in class, for instance.) It therefore suffices to maximize

$$f(x) = \frac{x}{x^2+ab}$$

on the domain $(0, \infty)$. The remainder of the solution is straightforward. We have

$$f'(x) = \frac{ab - x^2}{(x^2+ab)^2},$$

which vanishes when $x = \sqrt{ab}$ (we discard the other solution since it is not in the domain). Indeed, we have the following.

x	$\left(0, \sqrt{ab}\right)$	$\left(\sqrt{ab}, \infty\right)$
$f'(x)$	$+$	$-$
$f(x)$	increasing	decreasing

Thus the viewer should stand a distance $\sqrt{ab}$ from the wall.