ASSIGNMENT 2

Solutions

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1. Prove using the definition of limit that $\left\{\frac{\sin(2n)}{1+\sqrt{n}}\right\}$ converges.

   We claim that the sequence converges to 0. Since $|\sin(2n)| \leq 1$ for all n, it suffices to prove that $\left\{\left|\frac{\sin(2n)}{1+\sqrt{n}}\right|\right\} \leq \left\{\frac{1}{1+\sqrt{n}}\right\}$ converges to 0; that is, $\lim_{n\to\infty} \dfrac{1}{1+\sqrt{n}} = 0$.

   Let $\varepsilon > 0$ be given. Can we guarantee $\frac{1}{1+\sqrt{n}}$ to be within ε of 0 provided n is sufficiently large? We can. Indeed, $\frac{1}{\sqrt{n}}$ is guaranteed to be within ε of 0 provided $n > \frac{1}{\varepsilon^2}$, and since $0 < \frac{1}{1+\sqrt{n}} < \frac{1}{\sqrt{n}}$, this also guarantees $\frac{1}{1+\sqrt{n}}$ to be within ε of 0.

2. A line L is a *slant asymptote* of a curve C if the limit of the vertical distance between C and L at x is equal to 0 as x approaches ∞ or $-\infty$. For example, $y = x$ is a slant asymptote of the curve $y = x + \frac{1}{x}$.

   Find the slant asymptotes, if there are any, of the hyperbola $x^2 - y^2 = 1$. (If you propose any lines to be slant asymptotes, you must explain how you came up with the lines.)

   We write y explicitly in terms of x:
   $$y = \pm\sqrt{x^2 - 1}.$$
   If the line $y = mx + b$ is a slant asymptote, then one of the four limits
   $$\lim_{x\to\pm\infty} \left(\pm\sqrt{x^2-1} - (mx+b)\right) = \lim_{x\to\pm\infty} \frac{\left(1-m^2\right)x^2 + (2mb)x - \left(1+b^2\right)}{\pm\sqrt{x^2-1} + (mx+b)} \tag{1}$$
   must be equal to zero. To make these limits equal to zero, we consider constants m and b that make the numerator small enough to be dominated by the denominator; namely, $m = \pm 1$ and $b = 0$. Thus the lines $y = \pm x$ are good candidates for slant asymptotes.

   To prove that these lines are in fact slant asymptotes, we observe that all four limits given in (1) do in fact vanish when we take $m = \pm 1$ and $b = 0$:
   $$\lim_{x\to\pm\infty} \frac{-1}{\pm\sqrt{x^2-1} \pm x} = 0.$$

3. Given a sequence $\{a_n\}$, a subsequence $\{a_{n_k}\}$ is formed by selecting an infinite number of terms from that sequence (and preserving the order of terms). For example, $\{2, 3, 5, 7, 11, \ldots\}$ is a subsequence of $\{1, 2, 3, \ldots\}$. Prove that every bounded sequence has a convergent subsequence.

   (Hint. Suppose the sequence is bounded below by L and above by U. Bisect the interval $[L, U]$. At least one of the half-intervals must contain a subsequence; select a term from that half-interval. Then repeat the process.)

   We divide the interval $[L, U]$ in half. (At least) one of those halves contains infinitely many terms of the sequence. We call that half $[L_1, U_1]$ (if both halves have an infinite number of terms, we pick the "bottom" half, say) and select a term a_{n_1} in $[L_1, U_1]$.

Again, we divide $[L_1, U_1]$ in half, select a half $[L_2, U_2]$ with infinitely many terms of the sequence, and select a term a_{n_2} in $[L_2, U_2]$.

We repeat the process *ad infinitum*, getting a subsequence $\{a_{n_k}\}$ with terms in the sequence of nested intervals

$$[L_1, U_1] \supset [L_2, U_2] \supset [L_3, U_3] \supset \cdots,$$

respectively. $\{a_{n_k}\}$ converges. To see this, note that $\{L_k\}$ is an increasing, bounded sequence, which therefore converges to a number A. Similarly, $\{U_k\}$ is a decreasing, bounded sequence which converges to a number B. It remains only to show that $A = B$; this follows from the fact that $\{U_k - L_k\} = \left\{\frac{U-L}{2^k}\right\}$, which clearly converges to 0.