ASSIGNMENT 3 Solutions

1. Find the values of a for which $\sum_{n\geq 1} \frac{a^{n-1}}{2^n}$ converges. For those values, what does the series converge to?

 $\sum_{n\geq 1} \frac{a^{n-1}}{2^n} = \sum_{n\geq 1} \frac{1}{2} \left(\frac{a}{2}\right)^{n-1}$ is a geometric series which converges if (and only if) $\left|\frac{a}{2}\right| < 1$, or equivalently, -2 < a < 2. For such values, it converges to $\frac{\frac{1}{2}}{1-\frac{a}{2}} = \frac{1}{2-a}$.

2. When applying the Comparison and Limit Comparison Tests, it is often useful to compare series to *p*-series of the form $\sum_{n\geq 1} \frac{1}{n^p}$ where p > 0. However, none of our convergence tests so far allow us to determine the convergence of the *p*-series themselves.

In this question, we determine the convergence of *p*-series for $p \ge 2$ by proving a limited version of a result known as *Raabe's Test*. (It remains to determine the convergence for 1 , which we will do after we define the integral next term.)

Let
$$\sum_{n\geq 1} a_n$$
 be a series with all positive terms, and $\lim_{n\to\infty} n\left(1-\frac{a_{n+1}}{a_n}\right) = L > 1.$

(a) Let r be such that 1 < r < L. Prove that there exists some positive integer k such that

 $(r-1)a_n < (n-1)a_n - na_{n+1}$ for $n \ge k$.

(b) Let l > k be an integer. Using part (a), prove that

$$(r-1)(a_k + a_{k+1} + \dots + a_l) < (k-1)a_k.$$

(c) Using part (b) and the Monotone Convergence Theorem, prove that $\sum_{n>1} a_n$ converges.

(d) Using the result proven in the previous three parts, prove that $\sum_{n\geq 1} \frac{1}{n^p}$ converges for $p\geq 2$.

Since $\lim_{n \to \infty} n\left(1 - \frac{a_{n+1}}{a_n}\right) = L$, and since r < L, the terms on the left-hand side of the equation are eventually within L - r of L; equivalently, there exists a positive integer k such that, for $n \ge k$,

$$n\left(1 - \frac{a_{n+1}}{a_n}\right) > r$$

which after expanding, multiplying by a_n and subtracting a_n from both sides, is equivalent to

$$(r-1)a_n < (n-1)a_n - na_{n+1}.$$

This proves part (a).

By part (a), we have

$$\begin{array}{rcl} (r-1)a_k &< (k-1)a_k - ka_{k+1} \\ (r-1)a_{k+1} &< ka_{k+1} - (k+1)a_{k+2} \\ (r-1)a_{k+2} &< (k+1)a_{k+2} - (k+2)a_{k+3} \\ &\vdots \\ (r-1)a_{l-1} &< la_{l-1} - (l-1)a_l \\ (r-1)a_l &< (l+1)a_l - la_{l+1}. \end{array}$$

Taking the sum, which is telescoping on the right-hand side, we get

$$(r-1)(a_k + a_{k+1} + \dots + a_l) < (k-1)a_k - la_{l+1} < (k-1)a_k.$$

This proves part (b).

The partial sums of $\sum_{n\geq 1} a_n$ are clearly increasing. By part (b), the l^{th} partial sum has the bound

$$\sum_{n=1}^{l} a_n = a_1 + \dots + a_{k-1} + \sum_{n=k}^{l} a_n < a_1 + \dots + a_{k-1} + \left(\frac{k-1}{r-1}\right) a_k$$

Note that the right-hand side does not depend on l. Thus the partial sums are increasing and bounded, and the series converges. This proves part (c).

In order to show that $\sum_{n\geq 1} \frac{1}{n^p}$ converges for $p\geq 2$, we note that $\lim_{n\to\infty} n\left(1-\frac{\frac{1}{(n+1)^p}}{\frac{1}{n^p}}\right) = \lim_{n\to\infty} \frac{n(n+1)^p - n^{p+1}}{(n+1)^p}$ We use here the fact that $(n+1)^p = n^p + m^{p-1} + \dots + 1$ to conclude that

We use here the fact that $(n + 1)^p = n^p + pn^{p-1} + \dots + 1$ to conclude that

$$\lim_{n \to \infty} n \left(1 - \frac{\frac{1}{(n+1)^p}}{\frac{1}{n^p}} \right) = \lim_{n \to \infty} \frac{n \left(n^p + p n^{p-1} + \dots + 1 \right) - n^{p+1}}{n^p + p n^{p-1} + \dots + 1} = \lim_{n \to \infty} \frac{p n^p + \dots + n}{n^p + p n^{p-1} + \dots + 1} = p \ge 2,$$

ence $\sum_{n \ge 1} \frac{1}{n^p}$ converges.

- 3. On your UBC Blog, give one real-life example for each of the following: a function, a sequence and a series. Your examples should be described in plain English. Then answer the following questions.
 - (a) Does the graph of your function have a horizontal asymptote?
 - (b) Does your sequence converge?
 - (c) Does your series converge?

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Your post should be at least 150 words in length. It will be graded on clarity, originality and depth of understanding.

On your assignment submission, please include the URL of your blog.