ASSIGNMENT 4

Solutions

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1. Suppose
$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1.$$
Explain why it is false to draw any conclusions from this equality about the convergence of $\sum_{n\geq 1} a_n$.

   Let $a_n = \frac{1}{n}$; then $\sum_{n\geq 1} a_n$ is the harmonic series, which diverges. In this case,
$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{n}{n+1} = 1.$$
   If, on the other hand, $a_n = \frac{1}{n^2}$, then by Raabe's Test, which was proven on the previous assignment, $\sum_{n\geq 1} a_n$ converges. However, we still have
$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{n^2}{(n+1)^2} = 1.$$
   Therefore, the fact that the limit is equal to 1 gives no information about the convergence of the series.

2. Let $a_n = \begin{cases} \frac{n^2}{2^n} & \text{if } n \text{ is prime} \\ \frac{1}{n^3} & \text{otherwise} \end{cases}$. Determine if $\sum_{n\geq 1}(-1)^{n-1}a_n$ converges.

   We begin by observing that $\frac{1}{n^3} > \frac{n^2}{2^n}$ for all n sufficiently large. (One way to see this is to note the inequality may be rearranged to be $\frac{n}{\log(n)} > \frac{5}{\log(2)}$, and that the term on the left-hand side diverges to ∞ as n approaches ∞.) Since $\sum_{n\geq 1}\frac{1}{n^3}$ converges by Raabe's Test, the given series converges (absolutely).

3. (a) Explain how the terms in $\sum_{n\geq 1}\frac{(-1)^{n-1}}{n}$ may be rearranged so that the series converges to 2.

   (b) Explain how the terms in $\sum_{n\geq 1}\frac{(-1)^{n-1}}{n}$ may be rearranged so that the series diverges to ∞.

   Consider the subsequences $\{a_n\} = \left\{1, \frac{1}{3}, \frac{1}{5}, \ldots\right\}$ and $\{b_n\} = \left\{-\frac{1}{2}, -\frac{1}{4}, -\frac{1}{6}, \ldots\right\}$ of the sequence of terms in the given alternating harmonic series. We rearrange the series as follows.

   (a) Add enough terms from the beginning of $\{a_n\}$ so that the sum is "just larger" than 2 (that is, if we added one less term, our sum would be smaller than 2). Then remove those terms from $\{a_n\}$.

   (b) Add enough terms from the beginning of $\{b_n\}$ so that the sum is "just smaller" than 2. Then remove those terms from $\{b_n\}$.

   (c) Return to step (a).

The series then consists of strings of positive terms, say of lengths $r_1, r_2, r_3, \ldots$, alternating with strings of negative terms, say of lengths $s_1, s_2, s_3, \ldots$, with each string bringing the partial sums "just past" 2. We claim that this rearrangement converges to 2.

To see this, we note that the first r_1 terms bring the partial sum to within a_{r_1} of 2. The next s_1 terms bring the partial sum to within $-b_{s_1}$ of 2. The next r_2 terms bring the partial sum to within a_{r_2} of 2. The next s_2 terms bring the partial sum to within $-b_{s_2}$ of 2. The pattern continues *ad infinitum*.

Since the subsequences $\{a_n\}$ and $\{b_n\}$ both converge to 0, the difference between the partial sums of the rearrangement and 2 also converges to 0.