1. Find the points at which the graph of $f(x) = x^3 + 6x^2 + 12x - 10$ has a horizontal tangent line.

We have

$$f'(x) = 3x^2 + 12x + 12 = 3(x+2)^2,$$

which vanishes when x = -2. Thus the graph of $f(x) = x^3 + 6x^2 + 12^x - 10$ has a horizontal tangent line at (-2, -18).

2. Let

 $f(x) = \begin{cases} \frac{1}{q} & \text{if } x \text{ is a rational number } \frac{p}{q}, \text{ where } q > 0 \text{ and the number is written in lowest terms} \\ 0 & \text{if } x \text{ is an irrational number} \end{cases}$

(For example, $f\left(-\frac{3}{5}\right) = \frac{1}{5}$, and $f(\sqrt{2}) = 0$.) This is an example of a *nowhere differentiable* function. It is straightforward to show that f(x) is discontinuous, and therefore nondifferentiable, at all rational numbers x.

Prove that if a is an irrational number, then f(x) is not differentiable at a. (Hint: observe that for any positive integer q, there exists an integer p such that $\frac{p}{q} - \frac{1}{q} < a < \frac{p}{q}$. Select h such that $a + h = \frac{p}{q}$. Why does this mean f'(a) does not exist?)

By definition,

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} \frac{f(a+h)}{h}$$
(1)

if this limit exists. We shall show that this limit does not exist.

Consider the quotient

$$Q(h) = \lim_{h \to 0} \frac{f(a+h)}{h}.$$

There are arbitrarily small values of h for which a + h is irrational and Q(h) = 0. We shall show that the limit (1) does not exist by showing that there are also arbitrarily small values of h for which Q(h) > 1.

As in the hint, let $\frac{p}{q}$ be such that $\frac{p}{q} - \frac{1}{q} < a < \frac{p}{q}$ (in other words, $\frac{p}{q}$ is the smallest rational number with denominator q which is larger than a). Let h be such that $a + h = \frac{p}{q}$. Then $f(a+h) \ge \frac{1}{q}$ (it is larger if $\frac{p}{q}$ is not in lowest terms), but $h < \frac{1}{q}$. Thus Q(h) > 1. We are guaranteed that there exist arbitrarily small such values of h simply because there exist arbitrarily large choices of positive integers q.

Thus the limit (1) does not exist, and f(x) is not differentiable at a.

3. *Pascal's triangle* is an array of numbers whose first few rows are shown below. Every row begins and ends with 1. Every other entry is equal to the sum of the two entries immediately above it. The top row is called "row 0", the next row is called "row 1", and so on.

Let f(x) and g(x) be infinitely differentiable functions, and let F_n and G_n denote their n^{th} derivatives, respectively. (F_0 and G_0 denote f(x) and g(x), respectively.) In this question, you will prove the following claim: the n^{th} derivative of f(x)g(x) is equal to

$$a_n F_n G_0 + a_{n-1} F_{n-1} G_1 + \dots + a_1 F_1 G_{n-1} + a_0 F_0 G_n$$

where the coefficients $a_n, a_{n-1}, \ldots, a_1, a_0$ are the coefficients in row n of Pascal's triangle.

- (a) Prove that the claim is true in the case n = 1.
- (b) Prove that if the claim is true in the case n = k, then it is true in the case n = k + 1.
- (c) Explain carefully why parts (a) and (b) imply that the claim in true for all positive integers n.

The Product Rule states that the derivative of f(x)g(x) is $f'(x)g(x) + f(x)g'(x) = F_1G_0 + F_0G_1$. The coefficients, "1" and "1", are the numbers in row 1 of Pascal's triangle. This proves the claim in the case n = 1.

Now assume that the k^{th} derivative of f(x)g(x) is equal to

$$a_k F_k G_0 + a_{k-1} F_{k-1} G_1 + \dots + a_1 F_1 G_{k-1} + a_0 F_0 G_k,$$

where the coefficients $a_n, a_{n-1}, \ldots, a_1, a_0$ are the coefficients in row k of Pascal's triangle. Then the $(k+1)^{\text{th}}$ derivative is

$$(a_k F_{k+1}G_0 + a_k F_k G_1) + (a_{k-1}F_k G_1 + a_{k-1}F_{k-1}G_2) + (a_{k-2}F_{k-1}G_2 + a_{k-2}F_{k-2}G_3) + \cdots + (a_1F_2G_{k-1} + a_1F_1G_k) + (a_0F_1G_k + a_0F_0G_{k+1}).$$

Grouping the coefficients yields

But the coefficients a_k , $(a_k + a_{k-1})$, $(a_{k-1} + a_{k-2})$, ..., $(a_1 + a_0)$, a_0 are by definition the entries in row k + 1 of Pascal's triangle. Thus the claim is true in the case n = k + 1.

We proved directly that the claim is true in the case n = 1. By our work above, since the claim is true in the case n = 1, it is true in the case n = 2. Similarly, since the claim is true in the case n = 2, it is true in the case n = 3. This chain of logical implications may be extended indefinitely, to all positive integers n.