ASSIGNMENT 6

There are two parts to this assignment. The first part is on WeBWorK — the link is available on the course webpage. The second part consists of the questions on this page. You are expected to provide full solutions with complete justifications. You will be graded on the mathematical, logical and grammatical coherence and elegance of your solutions. Your solutions must be typed, with your name and student number at the top of the first page. If your solutions are on multiple pages, the pages must be stapled together.

*Your written assignment must be handed in **before your recitation on Friday, October 30**. The online assignment will close at **9:00 a.m. on Friday, October 30**.*

1. Find the points at which the graph of $f(x) = x^3 + 6x^2 + 12x - 10$ has a horizontal tangent line.

2. Let

$$f(x) = \begin{cases} \frac{1}{q} & \text{if } x \text{ is a rational number } \frac{p}{q}, \text{ where } q > 0 \text{ and the number is written in lowest terms} \\ 0 & \text{if } x \text{ is an irrational number} \end{cases}.$$

   (For example, $f\left(-\frac{3}{5}\right) = \frac{1}{5}$, and $f(\sqrt{2}) = 0$.) This is an example of a *nowhere differentiable* function. It is straightforward to show that $f(x)$ is discontinuous, and therefore nondifferentiable, at all rational numbers x.

   Prove that if a is an irrational number, then $f(x)$ is not differentiable at a. (Hint: observe that for any positive integer q, there exists an integer p such that $\frac{p}{q} - \frac{1}{q} < a < \frac{p}{q}$. Select h such that $a + h = \frac{p}{q}$. Why does this mean $f'(a)$ does not exist?)

3. *Pascal's triangle* is an array of numbers whose first few rows are shown below. Every row begins and ends with 1. Every other entry is equal to the sum of the two entries immediately above it. The top row is called "row 0", the next row is called "row 1", and so on.

$$1$$
$$1 \quad 1$$
$$1 \quad 2 \quad 1$$
$$1 \quad 3 \quad 3 \quad 1$$
$$1 \quad 4 \quad 6 \quad 4 \quad 1$$
$$1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1$$

   Let $f(x)$ and $g(x)$ be infinitely differentiable functions, and let F_n and G_n denote their n^{th} derivatives, respectively. (F_0 and G_0 denote $f(x)$ and $g(x)$, respectively.) In this question, you will prove the following claim: the nth derivative of $f(x)g(x)$ is equal to

$$a_n F_n G_0 + a_{n-1} F_{n-1} G_1 + \cdots + a_1 F_1 G_{n-1} + a_0 F_0 G_n,$$

   where the coefficients $a_n, a_{n-1}, \ldots, a_1, a_0$ are the coefficients in row n of Pascal's triangle.

   (a) Prove that the claim is true in the case $n = 1$.
   (b) Prove that if the claim is true in the case $n = k$, then it is true in the case $n = k + 1$.
   (c) Explain carefully why parts (a) and (b) imply that the claim in true for all positive integers n.