ASSIGNMENT 8

Solutions

1.

2. Find a polynomial of degree 5 agreeing with $f(x) = \sin(x)$ and its first 5 derivatives at $x = 0$.

Let $g(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5$. We have

$$\begin{array}{rcl} f(0) & = & 0 \\ f^{(1)}(0) & = & 1 \\ f^{(2)}(0) & = & 0 \\ f^{(3)}(0) & = & -1 \\ f^{(4)}(0) & = & 0 \\ f^{(5)}(0) & = & 1 \end{array} \qquad \begin{array}{rcl} g(x) & = & a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 \\ g^{(1)}(x) & = & a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + 5a_5x^4 \\ g^{(2)}(x) & = & 2a_2 + 6a_3x + 12a_4x^2 + 20a_5x^3 \\ g^{(3)}(x) & = & 6a_3 + 24a_4x + 60a_5x^2 \\ g^{(4)}(x) & = & 24a_4 + 120a_5x \\ g^{(5)}(x) & = & 120a_5 \end{array} \qquad \begin{array}{rcl} g(0) & = & a_0 \\ g^{(1)}(0) & = & a_1 \\ g^{(2)}(0) & = & 2a_2 \\ g^{(3)}(0) & = & 6a_3 \\ g^{(4)}(0) & = & 24a_4 \\ g^{(5)}(0) & = & 120a_5. \end{array}$$

In order for the functions and these derivatives to agree at $x = 0$, we take $a_0 = 0$, $a_1 = 1$, $a_2 = 0$, $a_3 = -\frac{1}{6}$, $a_4 = 0$ and $a_5 = \frac{1}{120}$; that is,

$$g(x) = x - \frac{1}{6}x^3 + \frac{1}{120}x^5 = x - \frac{x^3}{3!} + \frac{x^5}{5!}.$$

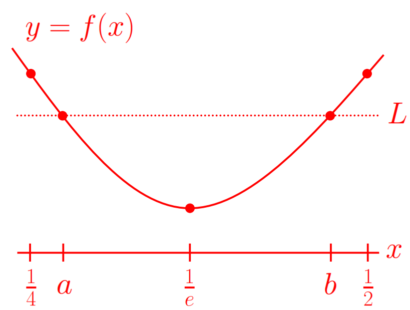
3. Note that $\left(\frac{1}{2}\right)^{1/2} = \left(\frac{1}{4}\right)^{1/4}$. Prove that there are infinitely many pairs of numbers $a < b$ such that $a^a = b^b$.

Let $f(x) = x^x$. We wish to prove that there exist infinitely many numbers a and b such that $f(a) = f(b)$.

Note that $f(x) = e^{x\log(x)}$, whence

$$f'(x) = e^{x\log(x)}\left(\log(x) + 1\right) = x^x(\log(x) + 1).$$

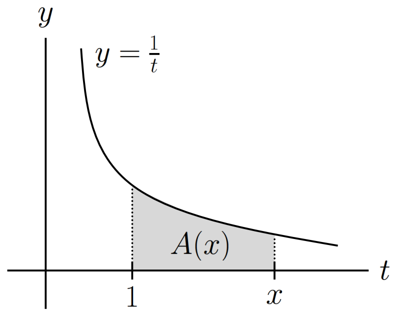
$f'(x)$ vanishes at $x = \frac{1}{e}$, with $f'(x) < 0$ for $0 < x < \frac{1}{e}$ and $f'(x) > 0$ for $\frac{1}{e} < x < \infty$. We conclude that $f(x)$ has a local minimum at $x = \frac{1}{e}$.



For any number L between $f\left(\frac{1}{4}\right)$ and $f\left(\frac{1}{e}\right)$, there exists a number a in $\left[\frac{1}{4}, \frac{1}{e}\right]$ such that $f(a) = L$ — this follows from the Intermediate Value Theorem (note we are assuming that $f(x)$ is continuous). Now L also lies between $f\left(\frac{1}{e}\right)$ and $f\left(\frac{1}{2}\right)$. Thus there exists a number b in $\left[\frac{1}{e}, \frac{1}{2}\right]$ such that $f(b) = L$. Thus $f(a) = f(b)$. This is illustrated in the picture above.

Since there are infinitely many choices for L, there are infinitely many pairs a and b such that $f(a) = f(b)$.

4. For positive x, let $A(x)$ be the area under the curve $y = \frac{1}{t}$ between $t = 1$ and $t = x$, as pictured below.

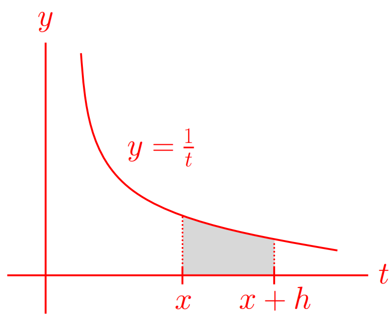


Let $\log(x) = \begin{cases} -A(x) & \text{if } 0 < x \le 1 \\ A(x) & \text{if } x > 1. \end{cases}$. Prove that $\frac{d}{dx}\log(x) = \frac{1}{x}$.
(Hint: use the definition of derivative, and find lower and upper bounds for $A(x)$.)

By the definition of derivative,

$$\frac{d}{dx}\log(x) = \lim_{h\to 0}\frac{\log(x+h) - \log(x)}{h}.$$

We have two cases: $h > 0$ and $h < 0$. Suppose $h > 0$. Then $\log(x+h) - \log(x)$ is equal to the area, shaded below, under the curve $y = \frac{1}{t}$ between $t = x$ and $t = x + h$.



It follows that

$$\frac{h}{x+h} < \log(x+h) - \log(x) < \frac{h}{x}.$$

Dividing by h and taking the limit yields

$$\lim_{h\to 0^+}\frac{\log(x+h) - \log(x)}{h} = \frac{1}{x}.$$

If on the other hand $h < 0$, then $\log(x) - \log(x+h)$ is equal to the area under the curve $y = \frac{1}{t}$ between $t = x + h$ and $t = x$. It follows that

$$-\frac{h}{x} < \log(x) - \log(x+h) < -\frac{h}{x+h}.$$

Dividing by $-h$ and taking the limit yields

$$\lim_{h\to 0^-}\frac{\log(x+h) - \log(x)}{h} = \frac{1}{x}.$$