ASSIGNMENT 2

Solutions

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1. Find a function $f(t)$ and a number a such that $1 + \int_a^x \frac{f(t)}{t^2}\,dt = 2\sqrt{x}$ for all $x > 0$.

   To find a, we let $x = a$, getting
   $$1 + \int_a^a \frac{f(t)}{t^2}\,dt = 1 = 2\sqrt{a};$$
   that is, $a = \frac{1}{4}$. To find $f(x)$, we differentiate the given equation, getting $\frac{f(x)}{x^2} = \frac{1}{\sqrt{x}}$; that is, $f(x) = x^{3/2}$.

2. Let $f(t)$ be continuous and $\int_l^r f(t)\,dt = 0$. Prove that there exists a point a such that $f(a) = 0$.

   Let $F(x) = \int_l^x f(t)\,dt$. Clearly $F(l) = 0$, and we are given that $F(r) = 0$. By the Fundamental Theorem of Calculus, $F(x)$ is differentiable everywhere, and therefore satisfies the conditions of the Mean Value Theorem (or Rolle's Theorem), from which we conclude that there exists a point a between l and r such that
   $$F'(a) = f(a) = 0.$$

3. In this problem, you will prove that π is irrational. Assume that π is rational; in particular, assume that $\pi = \frac{p}{q}$ where p and q are positive, coprime integers. Let
   $$f(t) = \frac{t^n(p - qt)^n}{n!}$$
   and
   $$F(t) = f(t) - f''(t) + f^{(4)}(t) - f^{(6)}(t) + \cdots + (-1)^n f^{(2n)}(t).$$
   A fact crucial to the proof is that $F(0) + F(\pi)$ is a positive integer. (You may use this fact without proof, but you can also prove it by expanding $f(t)$ using the Binomial Theorem and observing that $f^{(k)}(0)$ vanishes for $k < n$ and is an integer for $k \geq n$; it follows that $F(0)$ is a positive integer. The identity $f(\pi - t) = f(t)$ implies that $F(\pi)$ is also a positive integer. You may write out this proof in full for bonus marks.)

   (a) Prove that $\int_0^\pi f(t)\sin(t)\,dt \leq \frac{\pi^{n+1}p^n}{n!}$.

   (b) Prove that $\frac{d}{dt}\left(F'(t)\sin(t) - F(t)\cos(t)\right) = f(t)\sin(t)$.

   (c) Explain why it follows that π is irrational. (Hint: evaluate $\int_0^\pi f(t)\sin(t)\,dt$ directly.)

   On the interval $[0, \pi]$, we have $t \leq \pi$ and $p - qt \leq p$. This, along with the fact that $\sin(t) \leq 1$ on the same interval, implies that
   $$\int_0^\pi f(t)\sin(t)\,dt \leq \int_0^\pi \frac{\pi^n p^n}{n!}\,dt \leq \frac{\pi^{n+1}p^n}{n!}.$$
   This proves the inequality in part (a).

For part (b), we simply apply the Product Rule:

$$\begin{aligned}\tfrac{d}{dt}\left(F'(t)\sin(t) - F(t)\cos(t)\right) &= F''(t)\sin(t) + F'(t)\cos(t) - \left(F'(t)\cos(t) - F(t)\sin(t)\right)\\ &= \left(F''(t) + F(t)\right)\sin(t).\end{aligned}$$

Now

$$\begin{aligned}F''(t) &= f''(t) - f^{(4)}(t) + f^{(6)}(t) - f^{(8)}(t) + \cdots + (-1)^{n-1}f^{(2n)}(t) + (-1)^n f^{(2n+2)}(t)\\ &= f(t) - F(t) + (-1)^n f^{(2n+2)}(t).\end{aligned}$$

However, since $f(t)$ is a polynomial of degree $2n$, $f^{(2n+2)}(t) = 0$. Thus $F''(t) = f(t) - F(t)$, whence

$$\frac{d}{dt}\left(F'(t)\sin(t) - F(t)\cos(t)\right) = f(t)\sin(t).$$

We may therefore use the Fundamental Theorem of Calculus to evaluate that

$$\int_0^\pi f(t)\sin(t)\,dt = \left(F'(t)\sin(t) - F(t)\cos(t)\right)\Big|_0^\pi = F(\pi) + F(0),$$

which is a positive integer. However, it follows from part (a) and the fact that

$$\lim_{n\to\infty} \frac{\pi^{n+1}p^n}{n!} = 0$$

that $\displaystyle\int_0^\pi f(t)\sin(t)\,dt < 1$ for sufficiently large n. This is a contradiction. Therefore π is not rational.