ASSIGNMENT 2

There are two parts to this assignment. The first part is on WeBWorK — the link is available on the course webpage. The second part consists of the questions on this page. You are expected to provide full solutions with complete justifications. You will be graded on the mathematical, logical and grammatical coherence and elegance of your solutions. Your solutions must be typed, with your name and student number at the top of the first page. If your solutions are on multiple pages, the pages must be stapled together.

Your written assignment must be handed in ***before your recitation on Friday, January 22****. The online assignment will close at* ***9:00 a.m. on Friday, January 22****.*

1. Find a function $f(t)$ and a number a such that $1 + \displaystyle\int_a^x \frac{f(t)}{t^2}\,dt = 2\sqrt{x}$ for all $x > 0$.

2. Let $f(t)$ be continuous and $\displaystyle\int_l^r f(t)\,dt = 0$. Prove that there exists a point a such that $f(a) = 0$.

3. In this problem, you will prove that π is irrational. Assume that π is rational; in particular, assume that $\pi = \frac{p}{q}$ where p and q are positive, coprime integers. Let

$$f(t) = \frac{t^n(p - qt)^n}{n!}$$

and

$$F(t) = f(t) - f''(t) + f^{(4)}(t) - f^{(6)}(t) + \cdots + (-1)^n f^{(2n)}(t).$$

A fact crucial to the proof is that $F(0) + F(\pi)$ is a positive integer. (You may use this fact without proof, but you can also prove it by expanding $f(t)$ using the Binomial Theorem and observing that $f^{(k)}(0)$ vanishes for $k < n$ and is an integer for $k \geq n$; it follows that $F(0)$ is a positive integer. The identity $f(\pi - t) = f(t)$ implies that $F(\pi)$ is also a positive integer. You may write out this proof in full for bonus marks.)

(a) Prove that $\displaystyle\int_0^\pi f(t)\sin(t)\,dt \leq \frac{\pi^{n+1}p^n}{n!}$.

(b) Prove that $\frac{d}{dt}\left(F'(t)\sin(t) - F(t)\cos(t)\right) = f(t)\sin(t)$.

(c) Explain why it follows that π is irrational. (Hint: evaluate $\displaystyle\int_0^\pi f(t)\sin(t)\,dt$ directly.)