ASSIGNMENT 3

Solutions

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1. Let a and b be integers such that $|a| \neq |b|$. Evaluate $\displaystyle\int_{-\pi}^{\pi} \cos(at)\cos(bt)\,dt$.

   (Hint: use the formulas for $\cos(a+b)$ and $\cos(a-b)$ established last term to simplify the integrand.)

   We have the formulas
   $$\begin{aligned}\cos(a-b) &= \cos(a)\cos(b)+\sin(a)\sin(b),\\ \cos(a+b) &= \cos(a)\cos(b)-\sin(a)\sin(b).\end{aligned}$$
   Summing these and dividing by 2 yields the formula
   $$\cos(a)\cos(b) = \frac{1}{2}\left(\cos(a-b)+\cos(a+b)\right).$$
   Thus
   $$\int_{-\pi}^{\pi} \cos(at)\cos(bt)\,dt = \frac{1}{2}\int_{-\pi}^{\pi} \cos\left((a-b)t\right)\,dt + \frac{1}{2}\int_{-\pi}^{\pi} \cos\left((a+b)t\right)\,dt.$$
   For the first integral, we make the substitution $u=(a-b)t$. For the second integral, we make the substitution $v=(a+b)t$. Thus $du=(a-b)\,dt$, $dv=(a+b)dt$, and
   $$\begin{aligned}\int_{-\pi}^{\pi} \cos(at)\cos(bt)\,dt &= \frac{1}{2(a-b)}\int_{-(a-b)\pi}^{(a-b)\pi}\cos(u)\,du + \frac{1}{2(a+b)}\int_{-(a+b)\pi}^{(a+b)\pi}\cos(v)\,dv\\ &= \left.\frac{\sin(u)}{2(a-b)}\right|_{-(a-b)\pi}^{(a-b)\pi} + \left.\frac{\sin(v)}{2(a+b)}\right|_{-(a+b)\pi}^{(a+b)\pi}\\ &= 0.\end{aligned}$$

2. (a) Let $f(t)$ be continuous. Prove that $\displaystyle\int_0^r f(t)\,dt = \int_0^r f(r-t)\,dt$ for any r.

   (b) Prove that $\displaystyle\int_0^{\pi} \frac{t\sin(t)}{1+\cos^2(t)}\,dt = \frac{\pi^2}{4}$.

   To prove the identity in part (a), we make the substitution $u=r-t$, whence $du=-dt$ and
   $$\int_0^r f(r-t)\,dt = -\int_r^0 f(u)du = \int_0^r f(u)du.$$

   Applying this identity to the integral in part (b), we get
   $$\int_0^{\pi}\frac{t\sin(t)}{1+\cos^2(t)}\,dt = \int_0^{\pi}\frac{(\pi-t)\sin(t)}{1+\cos^2(t)}\,dt = \pi\int_0^{\pi}\frac{\sin(t)}{1+\cos^2(t)}\,dt - \int_0^{\pi}\frac{t\sin(t)}{1+\cos^2(t)}\,dt;$$
   that is,
   $$\int_0^{\pi}\frac{t\sin(t)}{1+\cos^2(t)}\,dt = \frac{\pi}{2}\int_0^{\pi}\frac{\sin(t)}{1+\cos^2(t)}\,dt. \tag{1}$$
   We make the substitution $u=\cos(t)$, whence $du=-\sin(t)\,dt$ and
   $$\int_0^{\pi}\frac{t\sin(t)}{1+\cos^2(t)}\,dt = -\int_1^{-1}\frac{1}{1+u^2}\,du = -\arctan(u)|_1^{-1} = \frac{\pi}{2}.$$
   The result then follows directly from (1).

3. When calculating integrals using the Fundamental Theorem of Calculus and the method of substitution, it is challenging to explain *why* a particular substitution is made, other than to say "It works". In this question, you are asked to distill the tactics you use to make a good substitution.

   On your UBC Blog, give at least three tips, or tactics, to make a good substitution. You may use examples, but your tips must not be specific to those examples only.

   Your post will be graded on clarity and the appropriateness of your tips.

   On your assignment submission, please include the URL of your blog. Then on Piazza, post the URL of your blog, along with your name.