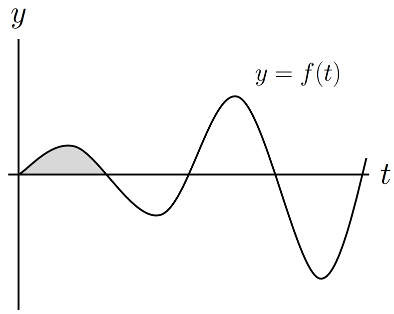
ASSIGNMENT 4

Solutions

1. Evaluate $\displaystyle\int_9^{10} \frac{\sqrt{t}}{t^2 - t}\,dt$.

   Let $u = \sqrt{t}$, whence $du = \frac{1}{2\sqrt{t}}\,dt = \frac{1}{2u}\,dt$ and
   $$\int_9^{10} \frac{\sqrt{t}}{t^2 - t}\,dt = 2\int_3^{\sqrt{10}} \frac{u^2}{u^4 - u^2}\,du = 2\int_3^{\sqrt{10}} \frac{1}{(u-1)(u+1)}\,du.$$
   Using integration by parts, we get
   $$\int_9^{10} \frac{\sqrt{t}}{t^2 - t}\,dt = \int_3^{\sqrt{10}} \left(\frac{1}{u-1} - \frac{1}{u+1}\right) du = \log\left(\frac{u-1}{u+1}\right)\Bigg|_3^{\sqrt{10}} = \log\left(\frac{\sqrt{10}-1}{\sqrt{10}+1}\right) - \log\left(\frac{1}{2}\right).$$

2. Let $f_a(t) = (\log(a))^{\log(a)}\, t\sin(at)$ be defined for all $a > 1$. The graph of this function is pictured below.



   Find a such that the shaded area under the first part of the curve is the smallest possible.

   The shaded area is given by
   $$\int_0^{\pi/a} f_a(t)\,dt = (\log(a))^{\log(a)} \int_0^{\pi/a} t\sin(at)\,dt.$$
   We evaluate the integral by parts, taking $u = t$ and $dv = \sin(at)\,dt$, whence $du = dt$, $v = -\frac{1}{a}\cos(at)$, and
   $$\int_0^{\pi/a} t\sin(at)\,dt = -\frac{t}{a}\cos(at)\Big|_0^{\pi/a} + \frac{1}{a}\int_0^{\pi/a}\cos(at)\,dt = -\frac{t}{a}\cos(at) + \frac{1}{a^2}\sin(at)\Big|_0^{\pi/a} = \frac{\pi}{a^2};$$
   that is,
   $$\int_0^{\pi/a} f_a(t)\,dt = (\log(a))^{\log(a)}\,\frac{\pi}{a^2}. \tag{1}$$
   It remains to minimize this with respect to a. We differentiate it with respect to a, getting
   $$\frac{d}{da}\left((\log(a))^{\log(a)}\,\frac{\pi}{a^2}\right) = \frac{\pi}{a^3}\,(\log(a))^{\log(a)}\,(\log(\log(a)) - 1)\,.$$
   ($\frac{d}{da}(\log(a))^{\log(a)} = \left(\frac{1}{a}\log(a)\right)^{\log(a)}(\log(\log(a)) - 1)$ may be calculated using implicit differentiation.) This vanishes, and indeed (1) is minimized, when $a = e^e$.

3. Let $f(t)$ be a function such that $f(t), f'(t), f''(t), \ldots, f^{(n+1)}(t)$ exist, and are continuous, on an interval containing a. Then for any x in that interval, *Taylor's Theorem (with integral remainder)* states that

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^n(a)}{n!}(x-a)^n + \frac{1}{n!}\int_a^x f^{(n+1)}(t)(x-t)^n\,dt. \quad (2)$$

In this question, you will prove and apply this theorem.

(a) Explain why equation (2) is true in the case $n = 0$; that is, explain why

$$f(x) = f(a) + \frac{1}{0!}\int_a^x f^{(0+1)}(t)(x-t)^0\,dt.$$

(Recall that we define $0! = 1$.)

(b) Suppose equation (2) is true in the case $n = k$. Prove that it is true in the case $n = k+1$. (Hint: use integration by parts on the last term.)

(c) Explain in one or two sentences why parts (a) and (b) imply that Taylor's Theorem is true.

By the Fundamental Theorem of Calculus,

$$f(a) + \frac{1}{0!}\int_a^x f^{(0+1)}(t)(x-t)^0\,dt = f(a) + \int_a^x f'(t)\,dt = f(a) + f(x) - f(a) = f(x).$$

This shows that (2) is true in the case $n = 0$.

Now suppose (2) is true in the case $n = k$; that is,

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^k(a)}{k!}(x-a)^k + \frac{1}{k!}\int_a^x f^{(k+1)}(t)(x-t)^k\,dt. \quad (3)$$

We evaluate the integral with the method of parts, taking $u = f^{(k+1)}(t)$ and $dv = (x-t)^k\,dt$, whence $du = f^{(k+2)}(t)\,dt$, $v = -\frac{(x-t)^{k+1}}{k+1}$, and

$$\begin{aligned}
\frac{1}{k!}\int_a^x f^{(k+1)}(t)(x-t)^k\,dt &= \frac{1}{k!}\left(-\frac{(x-t)^{k+1}}{k+1}f^{(k+1)}(t)\bigg|_a^x + \int_a^x \frac{(x-t)^{k+1}}{k+1}f^{(k+2)}(t)\,dt\right) \\
&= \frac{1}{k!}\left(\frac{f^{(k+1)}(a)}{k+1}(x-a)^{k+1} + \frac{1}{k+1}\int_a^x f^{(k+2)}(t)(x-t)^{k+1}\,dt\right) \\
&= \frac{f^{(k+1)}(a)}{(k+1)!}(x-a)^{k+1} + \frac{1}{(k+1)!}\int_a^x f^{(k+2)}(t)(x-t)^{k+1}\,dt.
\end{aligned}$$

(Note we assume here that $f^{(k+2)}(t)$ exists.) Substituting this back into (3) shows that (2) is true in the case $n = k+1$.

We proved directly in part (a) that

$$f(x) = f(a) + \frac{1}{0!}\int_a^x f^{(0+1)}(t)(x-t)^0\,dt.$$

In part (b), we proved that this implies

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{1}{1!}\int_a^x f^{(1+1)}(t)(x-t)^0\,dt,$$

which implies

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{1}{2!}\int_a^x f^{(2+1)}(t)(x-t)^0\,dt,$$

and so on until we get

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^n(a)}{n!}(x-a)^n + \frac{1}{n!}\int_a^x f^{(n+1)}(t)(x-t)^n\,dt.$$