ASSIGNMENT 5

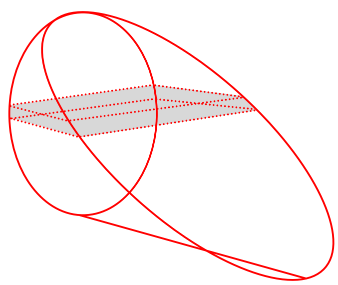
Solutions

1. A cylindrical glass of radius r and height h is filled with water and then tilted slowly until the remaining water covers the base of the glass exactly.

   (a) Calculate, using integration, the volume of the water remaining in the glass. (Hint: consider rectangular "slices" of water.)

   (b) Confirm your answer from part (a) using a geometric argument that does not require calculus.

   There are a number of ways to "slice" the water, but the suggested method is most straightforward. Picture the volume and slices of water as below.



   Consider axes imposed on the bottom of the cup and the side of the cup. The y-axes coincide in the two cases.

   The rectangle at y has width $2\sqrt{r^2 - y^2}$ and length $\frac{h}{2r}(r-y)$. Thus the cross-sectional area at y is $\frac{h}{r}(r-y)\sqrt{r^2-y^2}$, and the volume is

$$V = \int_{-r}^{r} \frac{h}{r}(r-y)\sqrt{r^2-y^2}\,dy = h\int_{-r}^{r}\sqrt{r^2-y^2}\,dy - \frac{h}{r}\int_{-r}^{r} y\sqrt{r^2-y^2}\,dy.$$

   For the first integral, let $y = r\sin(\theta)$, whence $dy = r\cos(\theta)\,d\theta$ and the first integral is equal to

$$r^2h\int_{-\pi/2}^{\pi/2}\cos^2(\theta)\,d\theta = \frac{r^2h}{2}\int_{-\pi/2}^{\pi/2}(1+\cos(2\theta))\,d\theta = \frac{r^2h}{2}\left(\theta + \frac{1}{2}\sin(2\theta)\right)\Bigg|_{-\pi/2}^{\pi/2} = \frac{\pi r^2 h}{2}.$$

The second integral is more straightforward; there, let $u = r^2 - y^2$, whence $du = 2y\,dy$ and the second integral is equal to

$$-\frac{h}{2r}\int_0^0 \sqrt{u}\,du = 0.$$

Thus $V = \frac{\pi r^2 h}{2}$. However, note that this answer may be recovered without any recourse to calculus. The portion of the cup filled with water is equivalent to the portion not filled with water (to see this, simply imagine rotating the portion filled with water). Thus the portion filled with water has volume equal to half the volume of the cup.

2. Work done against a force may be interpreted as storing up *potential energy*. Suppose work is done against a force $f(t)$ to move an object along the t-axis from $t = l$ to $t = r$. The potential energy stored is equal to

$$-\int_l^r f(t)\,dt.$$

For example, pulling a rope up to the top of a building stores potential energy which may later be released — for instance, as kinetic energy if the rope is thrown off the building. An object of mass m moving at a velocity v has kinetic energy $\frac{1}{2}mv^2$.

(a) The Earth exerts a gravitational force on an object of mass m (in kilograms) a height h (in metres) above the surface of the Earth of $\frac{km}{(r+h)^2}$, where k and r are positive constants (r is the radius of the Earth). Calculate the work done to raise a 1 kg object to an infinite height.

(b) The escape velocity of a projectile fired vertically from the surface of the Earth is the velocity necessary to guarantee that the projectile will not return to Earth. Calculate this velocity using your answer from part (a), along with Newton's second law of motion and the law of conservation of energy. You may ignore air resistance.

The work done to raise a 1 kg object to an infinite height is

$$\int_0^\infty \frac{km}{(r+h)^2}\,dh = \lim_t \to \infty \left.\frac{-km}{r+h}\right|_0^t = \frac{km}{r}.$$

By the conservation of energy, the escape velocity must be such that the projectile has kinetic energy sufficient to overcome the "gain" in potential energy; namely, we must have $\frac{1}{2}mv^2 \geq \frac{km}{r}$, or $v \geq \sqrt{\frac{2k}{r}}$. It remains to calculate k. This may be done by observing that, at the surface of the Earth, $\frac{km}{r^2} = mg$ by Newton's second law of motion; that is, $k = gr^2$. In summary, then, the escape velocity is $\sqrt{\frac{2k}{r}} = \sqrt{2gr}$, or roughly 11.2 km/s.

3. It is useful to revisit your midterm test to learn from it. A substantial body of research indicates that "reflection" is not only an effective learning tool, but one whose use distinguishes so-called "expert learners" from "novice learners". In this question, you are asked to reflect on your midterm.

On your UBC Blog, state in one or two paragraphs what you think was the most difficult question on the midterm, and why. (Note that a correct evaluation requires you to redo all of the questions on the midterm that you missed.) Then give at least two study tips to keep in mind for the final exam, which would have helped you on this midterm. Finally, post those study tips on Piazza, along with the URL of your blog.

On your assignment submission, please include the URL of your blog.