# ASSIGNMENT 6

Solutions

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1. Find a power series representation for the function $f(x) = \dfrac{5x}{(x-2)^2}$.

   We being with the geometric series

$$\frac{1}{2-x} = \frac{1}{2}\left(\frac{1}{1-\frac{x}{2}}\right) = \frac{1}{2}\sum_{n\geq 0}\left(\frac{x}{2}\right)^n.$$

   Differentiating yields

$$\frac{1}{(2-x)^2} = \frac{1}{(x-2)^2} = \frac{1}{4}\sum_{n\geq 1} n\left(\frac{x}{2}\right)^{n-1}.$$

   Finally, we multiply by $5x$ to get

$$\frac{5x}{(x-2)^2} = \frac{5x}{4}\sum_{n\geq 1} n\left(\frac{x}{2}\right)^{n-1} = \sum_{n\geq 1}\frac{5n}{2^{n+1}}x^n = \frac{5}{4}x + \frac{10}{8}x^2 + \frac{15}{16}x^3 + \cdots.$$

   Note that the power series representation may also be derived by squaring the power series for $\frac{1}{x-2}$ and then multiplying by $5x$.

2. The *Fibonacci sequence* $\{F_n\} = \{0, 1, 1, 2, 3, 5, \ldots\}$ is defined by the recurrence relation

$$F_n = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ F_{n-1} + F_{n-2} & \text{otherwise} \end{cases}.$$

   Let $f(x) = \sum_{n\geq 0} F_n x^n$. In this question we find a closed-form expression for the Fibonacci numbers.

   (a) Prove that $f(x) = \dfrac{x}{1-x-x^2}$. (Hint: consider the series $f(x)$, $xf(x)$ and $x^2 f(x)$.)

   (b) Find the partial fraction decomposition of $\dfrac{x}{1-x-x^2}$.

   (c) Use your answer from part (b) to prove that $F_n = \dfrac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$.

   We have

$$\begin{aligned} f(x) &= F_0 + F_1 x + F_2 x^2 + F_3 x^3 + \cdots \\ xf(x) &= F_0 x + F_1 x^2 + F_2 x^3 + F_3 x^4 + \cdots \\ x^2 f(x) &= F_0 x^2 + F_1 x^3 + F_2 x^5 + F_3 x^6 + \cdots. \end{aligned}$$

   Subtracting the second and third lines from the first, we get

$$f(x)\left(1-x-x^2\right) = F_0 + (F_1 - F_0)\,x + (F_2 - F_1 - F_0)\,x^2 + (F_3 - F_2 - F_1)\,x^3 + \cdots = x$$

   — the last equality follows from the recurrence relation, as well as the fact that $F_0 = 0$ and $F_1 = 1$. Thus

$$f(x) = \frac{x}{1-x-x^2}. \tag{1}$$

We now apply a partial fraction decomposition to the right-hand side. Let $r_1 = \frac{-1+\sqrt{5}}{2}$ and $r_2 = \frac{-1-\sqrt{5}}{2}$. Then we set

$$\frac{x}{1-x-x^2} = \frac{-x}{(x-r_1)(x-r_2)} = \frac{A}{x-r_1} + \frac{B}{x-r_2} = \frac{(A+B)x - (Ar_2 + Br_1)}{(x-r_1)(x-r_2)}. \tag{2}$$

Comparing coefficients, we get the system of equations

$$\begin{aligned} A + B &= -1 \\ Ar_2 + Br_1 &= 0, \end{aligned}$$

from which it follows that

$$A = \frac{r_1}{r_2 - r_1} = \frac{1-\sqrt{5}}{2\sqrt{5}} \quad \text{and} \quad B = \frac{-r_2}{r_2 - r_1} = \frac{-1-\sqrt{5}}{2\sqrt{5}}.$$

Thus from (1) and (2), we have

$$f(x) = \frac{-A}{r_1 - x} + \frac{-B}{r_2 - x} = \frac{-A}{r_1}\left(\frac{1}{1-\frac{x}{r_1}}\right) + \frac{-B}{r_2}\left(\frac{1}{1-\frac{x}{r_2}}\right) = \frac{-A}{r_1}\sum_{n\geq 0}\left(\frac{x}{r_1}\right)^n + \frac{-B}{r_2}\sum_{n\geq 0}\left(\frac{x}{r_2}\right)^n.$$

Combining the series, we get

$$f(x) = \sum_{n\geq 0}\left(\frac{-A}{r_1^{n+1}} + \frac{-B}{r_2^{n+1}}\right)x^n = \sum_{n\geq 0}\frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)x^n$$

as needed.

3. The *binomial theorem* states that for a positive integer $n$, the function $(1+x)^n$ has the power series representation

$$(1+x)^n = \sum_{k=0}^{n}\binom{n}{k}x^k$$

where

$$\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!}.$$

In this question we extend the theorem to all real numbers. Let $t$ be a number which is not a positive integer or zero, and define

$$\binom{t}{k} = \begin{cases} \frac{t(t-1)\cdots(t-k+1)}{k!} & \text{if } k \neq 0 \\ 1 & \text{if } k = 0 \end{cases}.$$

(a) Verify that $(k+1)\binom{t}{k+1} + k\binom{t}{k} = t\binom{t}{k}$.

(b) Let $f(x) = \sum_{k\geq 0}\binom{t}{k}x^k$. Use the Ratio Test to show that the series converges for $|x| < 1$.

(c) Prove that $(1+x)f'(x) = tf(x)$ for $|x| < 1$.

(d) Prove that $\frac{d}{dx}\frac{f(x)}{(1+x)^t} = 0$, and explain why it follows that $(1+x)^t = \sum_{k=0}^{\infty}\binom{t}{k}x^k$.

We have

$$\begin{aligned}
(k+1)\binom{t}{k+1} + k\binom{t}{k} &= (k+1)\frac{t(t-1)\cdots(t-k+1)(t-k)}{(k+1)!} + k\frac{t(t-1)\cdots(t-k+1)}{k!} \\
&= \frac{t(t-1)\cdots(t-k+1)(t-k)}{k!} + k\frac{t(t-1)\cdots(t-k+1)}{k!} \\
&= \frac{t(t-1)\cdots(t-k+1)}{k!}(t-k+k) \\
&= \tbinom{t}{k}t.
\end{aligned}$$

For part (b), we observe that

$$\begin{aligned}
\lim_{k\to\infty}\left|\frac{\binom{t}{k+1}x^{k+1}}{\binom{t}{k}x^k}\right| &= \lim_{k\to\infty}\left|\frac{t(t-1)\cdots(t-k+1)(t-k)}{(k+1)!}\frac{k!}{t(t-1)\cdots(t-k+1)}\right||x| \\
&= \lim_{k\to\infty}\left|\frac{t-k}{k+1}\right||x| \\
&= |x|.
\end{aligned}$$

Thus $f(x) = \sum_{k\geq 0}\binom{t}{k}x^k$ converges for $|x|<1$. Note we require here that $\binom{t}{k}\neq 0$, which is true since $t$ is not a positive integer or zero.

Differentiating term-by-term,

$$\begin{aligned}
(1+x)f'(x) &= (1+x)\sum_{k\geq 1}k\binom{t}{k}x^{k-1} \\
&= (1+x)\sum_{k\geq 0}(k+1)\binom{t}{k+1}x^k \\
&= \sum_{k\geq 0}(k+1)\binom{t}{k+1}x^k + \sum_{k\geq 1}k\binom{t}{k}x^k \qquad (3) \\
&= t + \sum_{k\geq 1}\left((k+1)\binom{t}{k+1} + k\binom{t}{k}\right)x^k \\
&= t + \sum_{k\geq 1}t\binom{t}{k}x^k \\
&= tf(x).
\end{aligned}$$

Finally, by the Quotient Rule we have

$$\frac{d}{dx}\frac{f(x)}{(1+x)^t} = \frac{(1+x)^t f'(x) + t(1+x)^{t-1}f(x)}{(1+x)^{2t}} = \frac{(1+x)f'(x)+tf(x)}{(1+x)^{t+1}} = 0,$$

with the last equality following from (3). This implies that $\frac{f(x)}{(1+x)^t}$ is a constant: say $f(x) = a(1+x)^t$ for some number $a$. Since $f(0)=1$, it follows that $a=1$ and $f(x) = (1+x)^t$ as conjectured.