ASSIGNMENT 6

There are two parts to this assignment. The first part is on WeBWorK — the link is available on the course webpage. The second part consists of the questions on this page. You are expected to provide full solutions with complete justifications. You will be graded on the mathematical, logical and grammatical coherence and elegance of your solutions. Your solutions must be typed, with your name and student number at the top of the first page. If your solutions are on multiple pages, the pages must be stapled together.

*Your written assignment must be handed in **before your recitation on Friday, March 11**. The online assignment will close at **9:00 a.m. on Friday, March 11**.*

1. Find a power series representation for the function $f(x) = \dfrac{5x}{(x-2)^2}$.

2. The *Fibonacci sequence* $\{F_n\} = \{0, 1, 1, 2, 3, 5, \ldots\}$ is defined by the recurrence relation

$$F_n = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ F_{n-1} + F_{n-2} & \text{otherwise} \end{cases}.$$

   Let $f(x) = \displaystyle\sum_{n \geq 0} F_n x^n$. In this question we find a closed-form expression for the Fibonacci numbers.

   (a) Prove that $f(x) = \dfrac{x}{1 - x - x^2}$. (Hint: consider the series $f(x)$, $xf(x)$ and $x^2 f(x)$.)

   (b) Find the partial fraction decomposition of $\dfrac{x}{1 - x - x^2}$.

   (c) Use your answer from part (b) to prove that $F_n = \dfrac{1}{\sqrt{5}} \left(\left(\frac{1 + \sqrt{5}}{2} \right)^n - \left(\frac{1 - \sqrt{5}}{2} \right)^n \right)$.

3. The *binomial theorem* states that for a positive integer n, the function $(1 + x)^n$ has the power series representation

$$(1 + x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

   where

$$\binom{n}{k} = \frac{n(n-1) \cdots (n-k+1)}{k!}.$$

   In this question we extend the theorem to all real numbers. Let t be a number which is not a positive integer or zero, and define

$$\binom{t}{k} = \begin{cases} \frac{t(t-1) \cdots (t-k+1)}{k!} & \text{if } k \neq 0 \\ 1 & \text{if } k = 0 \end{cases}.$$

   (a) Verify that $(k + 1) \dbinom{t}{k+1} + k \dbinom{t}{k} = t \dbinom{t}{k}$.

   (b) Let $f(x) = \displaystyle\sum_{k \geq 0} \binom{t}{k} x^k$. Use the Ratio Test to show that the series converges for $|x| < 1$.

   (c) Prove that $(1 + x)f'(x) = tf(x)$ for $|x| < 1$.

   (d) Prove that $\dfrac{d}{dx} \dfrac{f(x)}{(1 + x)^t} = 0$, and explain why it follows that $(1 + x)^t = \sum_{k=0}^{\infty} \binom{t}{k} x^k$.