ASSIGNMENT 7

Solutions

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1. (a) Use a linear approximation to estimate $\log(1.01)$.
   (b) Is your approximation $L(1.01)$ an underestimate, an overestimate, or equal to $\log(1.01)$? Justify your answer.
   (c) Bound the size of the error $E(1.01) = \log(1.01) - L(1.01)$ using an appropriate theorem (and not using a calculator).

   We approximate $\log(x)$ about 1, getting the linearization

   $$L(x) = \log(1) + \frac{1}{1}(x-1) = x - 1;$$

   that is, $\log(1.01) \approx 0.01$. This is an overestimate, since $\log(x)$ is concave down on its domain (hence its tangent lines lie above it, except at the points of tangency). Using the error formula for linear approximations, the error is given by

   $$|E(1.01)| = \left| \frac{-\frac{1}{s^2}}{2}(1.01-1)^2 \right| = \frac{1}{20000s^2}$$

   for some s between 1 and 1.01. We take $s = 1$ to get an upper bound $\frac{1}{20000}$ on the error.

2. Let $f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$.

   (a) Prove that $f^{(n)}(0) = 0$ for all n.
   (b) Explain why the Maclaurin series for $f(x)$ does not converge to $f(x)$ on any open interval.

   We shall prove that $f^{(n)}(0) = 0$ for all n by induction on n.

   First,

   $$f'(0) = \lim_{h\to 0} \frac{f(h) - f(0)}{h} = \lim_{h\to 0} \frac{e^{-1/h^2}}{h} = \lim_{h\to 0} \frac{1/h}{e^{1/h^2}} = 0,$$

   with the last equality following from L'Hospital's Rule. This proves the base case.

   Next, we assume that $f^{(k)}(0) = 0$. Then

   $$f^{(k+1)}(0) = \lim_{h\to 0} \frac{f^{(k)}(h) - f^{(k)}(0)}{h} = \lim_{h\to 0} \frac{f^{(k)}(h)}{h}. \tag{1}$$

   We turn our attention to the numerator $f^{(k)}(h)$. Note that

   $$f'(h) = \frac{2}{h^3} e^{-1/h^2} = P_1\left(\tfrac{1}{h}\right) e^{-1/h^2},$$

   where $P_1(x) = 2x^3$ is a polynomial. In fact, if

   $$f^{(j)}(h) = P_j\left(\tfrac{1}{h}\right) e^{-1/h^2}$$

where $P_j(x)$ is a polynomial, then

$$f^{(j+1)}(h) = P_j'\left(\tfrac{1}{h}\right)\left(-\frac{1}{h^2}\right)e^{-1/h^2} + P_j\left(\tfrac{1}{h}\right)P_1\left(\tfrac{1}{h}\right)e^{-1/h^2} = P_{j+1}\left(\tfrac{1}{h}\right)e^{-1/h^2},$$

where $P_{j+1}(x)$ is a polynomial. Thus

$$f^{(k)}(h) = P_k\left(\tfrac{1}{h}\right)e^{-1/h^2}$$

where $P_k(x)$ is a polynomial; and returning to (1),

$$f^{(k+1)}(0) = \lim_{h\to 0}\frac{1}{h}P_k\left(\tfrac{1}{h}\right)e^{-1/h^2}.$$

To see that this vanishes, note that, if it is expanded, each term of $\frac{1}{h}P_k\left(\frac{1}{h}\right)e^{-1/h^2}$ is of the form

$$a\frac{e^{-1/h^2}}{h^m} = a\frac{1/h^m}{e^{1/h^2}}$$

for a constant a and positive integer m. An appropriate number of applications of L'Hospital's Rule shows that the limit of this vanishes as $h \to 0$.

Thus $f^{(n)}(0) = 0$ for all n, and the Maclaurin series for $f(x)$ is identically 0. However, since $f(x) \neq 0$ except at 0, the the Maclaurin series for $f(x)$ does not converge to $f(x)$ on any open interval.

3. In the previous question, you proved that the given function is not *analytic* at 0. On the other hand, the function e^x *is* analytic at 0 (and indeed everywhere). On your UBC Blog, explain in one or two paragraphs, with the use of examples, what it means for a function to be *analytic* at c, and why this is a useful definition. In particular, what distinguishes a function that is infinitely differentiable from one that is analytic?

   Note that you will likely begin with a search on Wikipedia, Google, or in the index of a textbook, but that you must paraphrase what you find in a way that is comprehensible to someone with your background in calculus. You should avoid jargon and focus on the logical structure of what you write.

   On your assignment submission, please include the URL of your blog.