ASSIGNMENT 8

Solutions

1. Determine the Maclaurin polynomial of minimal degree needed to calculate $\sin(1)$ to an accuracy of 3 decimal places.

   The Maclaurin series for $\sin(x)$ is

   $$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n\geq 1} \frac{(-1)^{n-1}x^{2n-1}}{(2n-1)!}.$$

   The error $E(1) = \sin(1) - M_{2n-1}(1)$, where $M_{2n-1}(x)$ is the degree $2n-1$ Maclaurin polynomial, satisfies

   $$|E(1)| = \left|\frac{(-1)^n s^{2n+1}}{(2n+1)!}\right| \leq \frac{1}{(2n+1)!}$$

   where s is a number between 0 and 1. We wish to have $|E(1)| \leq 0.001$, which is satisfied for $n = 3, 4, 5, \ldots$ (and not for $n = 1, 2$). Thus the Maclaurin polynomial of degree 5 gives the necessary accuracy: $1 - \frac{1}{3!} + \frac{1}{5!}$ is an accurate approximation of $\sin(1)$ to 3 decimal places.

2. Let $f(x) = (x-1)^7 e^x$. Calculate $f^{(50)}(1)$.

   Note that $f(x) = e(x-1)^7 e^{x-1}$ has Taylor series

   $$e(x-1)^7 \sum_{n\geq 0} \frac{(x-1)^n}{n!} = \sum_{n\geq 0} \frac{e(x-1)^{n+7}}{n!} = \sum_{n\geq 0} \frac{f^{(n)}(1)}{n!}(x-1)^n.$$

   Comparing coefficients yields $\frac{f^{(50)}(1)}{50!} = \frac{e}{43!}$; that is, $f^{(50)}(1) = \frac{50!}{43!}e$.

3. A random variable is *normally distributed* with mean μ and standard deviation $\sigma > 0$ if its *probability density function* is given by

   $$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/\left(2\sigma^2\right)}$$

   The probability that a normally distributed random variable is in the interval (l, r) is given by $\displaystyle\int_l^r f(t)\,dt$.

   Let P be the probability that a normally distributed random variable is "within one standard deviation of the mean" — that is, in the interval $(\mu - \sigma, \mu + \sigma)$. Calculate P to within 0.001 of its actual value.

   We wish to calculate

   $$P = \frac{1}{\sigma\sqrt{2\pi}} \int_{\mu-\sigma}^{\mu+\sigma} e^{-(t-\mu)^2/\left(2\sigma^2\right)}\,dt.$$

   We make the substitution $u = \frac{t-\mu}{\sigma}$, whence $du = \frac{1}{\sigma}\,dt$ and

   $$P = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{-u^2/2}\,du = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} \sum_{n\geq 0} \frac{\left(-u^2/2\right)^n}{n!}\,du.$$

Since the series converges everywhere, we may integrate it term-by-term, getting

$$P = \frac{1}{\sqrt{2\pi}} \sum_{n\geq 0} \int_{-1}^{1} \frac{\left(-u^2/2\right)^n}{n!}\, du = \frac{1}{\sqrt{2\pi}} \sum_{n\geq 0} \frac{(-1)^n}{2^n n!} \int_{-1}^{1} u^{2n}\, du = \frac{1}{\sqrt{2\pi}} \sum_{n\geq 0} \frac{(-1)^n}{2^{n-1}(2n+1)n!}. \tag{1}$$

This is a convergent alternating series. In order to achieve the desired accuracy, we observe that, for any alternating series $\sum_{n\geq 0}(-1)^n a_n$ that converges to L, say, the difference between L and the n^{th} partial sum S_n satisfies

$$|L - S_n| \leq |S_{n+1} - S_n| = a_{n+1},$$

since L lies between successive partial sums. In other words, the error is "no worse than the next term in the series". Thus if we take P_n to be the first n terms of the series in (1), the error $E_n = P - P_n$ satisfies

$$|E_n| \leq \frac{1}{\sqrt{2\pi}2^n(2n+3)(n+1)!}.$$

We have $|E_n| \leq 0.001$ if $n = 3, 4, 5, \ldots$ (and not if $n = 0, 1, 2$). Thus

$$\frac{1}{\sqrt{2\pi}} \sum_{n=0}^{3} \frac{(-1)^n}{2^{n-1}(2n+1)n!} = \frac{1}{\sqrt{2\pi}} \left(2 - \frac{1}{3} + \frac{1}{20} - \frac{1}{168}\right) \approx 0.6825$$

approximates P to within 0.001 of its actual value.