ASSIGNMENT 9

Solutions

---

1. The *Gompertz differential equation* is a version of the logistic differential equation also used to model population growth. It is
$$\frac{dP}{dt} = rP\log\left(\frac{K}{P}\right)$$
where r and K are constants. Solve the differential equation, assuming $r = 0.05$, $K = 1000$ and the initial condition $P(0) = 500$. Then sketch the graph of your solution. (You do not have to justify the shape of the graph.)

   This equation is separable: we have
$$\int \frac{1}{P\log\left(\frac{K}{P}\right)}\,dP = \int r\,dt.$$
   Let $u = \log\left(\frac{K}{P}\right)$, whence $du = -\frac{1}{P}\,dP$ and
$$\int \frac{1}{P\log\left(\frac{K}{P}\right)}\,dP = -\int \frac{1}{u}\,du = -\log|u| = -\log\left|\log\left(\frac{K}{P}\right)\right|.$$
   (We absorb the constant obtained from antidifferentiation into the right-hand side, below.) Thus
$$\begin{aligned} -\log\left|\log\left(\tfrac{K}{P}\right)\right| &= rt + c \\ \log\left(\tfrac{K}{P}\right) &= \pm e^{-rt-c} \end{aligned}$$
   Setting $t = 0$ yields $\log(2) = \pm e^{-c}$; that is, $c = -\log(\log(2))$ and
$$\begin{aligned} \log\left(\tfrac{K}{P}\right) &= \log(2)e^{-0.05t} \\ \tfrac{K}{P} &= 2^{e^{-0.05t}} \\ P &= \frac{K}{2^{e^{-0.05t}}}. \end{aligned}$$

2. The *Bessel equation* is a differential equation which arises in problems of wave propagation. It is given by
$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2-1)y = 0. \tag{1}$$
   In this question, you will find one nontrivial solution to the Bessel equation using power series.

   (a) Let $y = \sum_{n\geq 0} a_n x^{\mu+n}$ be a solution to (1). Substitute this into the left-hand side of the Bessel equation, and simplify into a single power series. (Your answer should use the summation notation "$\sum$", but you may find it helpful to write the first two terms separately.)

   (b) Let $a_0 = 1$. Find a pattern to describe $a_1, a_2, a_3, a_4, \ldots$.

   (c) Prove that the series y, with the coefficients as described in part (b), converges for all x.

We have

$$\frac{dy}{dx} = \sum_{n\geq 0} (\mu+n)\, a_n x^{\mu+n-1} \quad \text{and} \quad \frac{d^2y}{dx^2} = \sum_{n\geq 0} (\mu+n)\,(\mu+n-1)\, a_n x^{\mu+n-2}.$$

The left-hand side of (1) may be rewritten

$$\begin{aligned} & \sum_{n\geq 0} \left((\mu+n)\,(\mu+n-1)\, a_n x^{\mu+n} + (\mu+n)\, a_n x^{\mu+n} + a_n x^{\mu+n+2} - a_n x^{\mu+n}\right) \\ = & \sum_{n\geq 0} \left(((\mu+n)\,(\mu+n-1) + (\mu+n) - 1)\, a_n x^{\mu+n} + a_n x^{\mu+n+2}\right) \\ = & \sum_{n\geq 0} \left(\left((\mu+n)^2 - 1\right) a_n x^{\mu+n} + a_n x^{\mu+n+2}\right). \end{aligned}$$

We divide both sides of (1) by x^μ to get

$$\begin{aligned} 0 &= \sum_{n\geq 0} \left(\left((\mu+n)^2 - 1\right) a_n x^n + a_n x^{n+2}\right) \\ &= \left(\mu^2 - 1\right) a_0 + \left((\mu+1)^2 - 1\right) a_1 x + \sum_{n\geq 2} \left(\left((\mu+n)^2 - 1\right) a_n + a_{n-2}\right) x^n. \end{aligned} \tag{2}$$

Since this must be identically 0 — that is, all of the coefficients must vanish — and since $a_0 = 1$, we must have, from the first two terms,

$$\mu^2 - 1 = 0 \tag{3}$$

and

$$\left((\mu+1)^2 - 1\right) a_1 = 0. \tag{4}$$

From (3) we take $\mu = 1$, which means by (4) that $a_1 = 0$. From the remaining terms of (2), we get the recurrence relation

$$a_n = -\frac{a_{n-2}}{(\mu+n)^2 - 1} \tag{5}$$

for $n \geq 2$. Thus $a_3 = a_5 = a_7 = \cdots = 0$, and

$$a_2 = -\frac{1}{8}, a_4 = \frac{1}{8\cdot 24}, a_6 = -\frac{1}{8\cdot 24\cdot 48}, a_8 = \frac{1}{8\cdot 24\cdot 48\cdot 80}, \ldots.$$

This may be written as

$$a_{2n} = \frac{(-1)^n}{2^{2n}(n+1)!n!}.$$

Thus one solution to (1) is

$$y = \sum_{n\geq 0} \frac{(-1)^n}{2^{2n}(n+1)!n!} x^{2n+1}.$$

This converges everywhere, by the Ratio Test.

3. In the first assignment of MATH 100, we stated that the exercise of posting mathematical reflections on your blog was to "train your ability to explain abstract mathematical ideas, a key skill in higher-level mathematics".

   Pick any mathematical idea you learned about in MATH 100 or MATH 101, from limits to differential equations, or even a precalculus idea that was used. Then explain it on your blog in one to three paragraphs. Imagine you are introducing someone to the idea for the first time. Your explanation should be in plain English, it should avoid jargon and mathematical notation, and it must be different from any explanation that we gave to you in MATH 100 or MATH 101.

   On your assignment submission, please include the URL of your blog.