An introduction to, wait for it...the Renewal Process

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Abstract

A renewal process counts the number of times that a system returns to its initial state, called a renewal. Renewal theory is the study of the waiting times and long-term behaviour of how these renewals occur. In this paper we begin by discussing the general theory of renewal processes and explore the distribution renewal counting process. We discuss the key renewal theorem and its implications. We also give variants of the renewal processes, including renewal reward, alternating renewal, and delayed renewal processes. Finally we explore in detail the inspection paradox and the distribution of waiting times.

1 Introduction

Many complicated processes have randomly occurring instances where the processes returns to a state that is probabilistically equivalent to the initial state. When the system returns to the initial state, we say that a renewal has occurred. The renewal process counts the number of such renewals over a period of time. Renewal theory is the study of how these renewals occur, when they occur, and the limiting behaviour of the system. It turns out that in the limit, all the renewal process satisfy some fairly stringent conditions, and the limiting behaviour can be shown to be completely determined by the distribution of the time between successive renewals.

We will explore analogies of the strong law of large numbers and central limit theorem for renewal processes. We will then discuss the key renewal theorem, and variants of the renewal process including reward renewal process, alternating renewal process, and delayed renewal process. Finally, we will give a in-depth analysis of waiting times and the inspection paradox.

2 Mathematical preliminaries

There are certain theorems/concepts that will be used throughout the paper that have not been covered in lectures but have been covered in either a measure theory course or a probability course. Therefore these results will be listed here without proof. Throughout the paper we will assume the underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$ where Ω is the underlying set, $\mathcal{F}$ is the sigma algebra of subsets of Ω representing the possible outcomes, and $\mathbb{P}$ is a probability measure.

2.1 Measure theory

There are certain facts from measure theory that we will use without proof throughout the paper.

Theorem 2.1 (Monotone Convergence Theorem (MCT)). *[Roy68] Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of random variables, that are increasing and converge pointwise to to some some X (with probability 1). Then*

$$\lim_{n\to\infty}\langle X_n\rangle = \langle \lim_{n\to\infty} X_n\rangle = \langle X\rangle.$$

Theorem 2.2 (Lebesgue Dominated Convergence Theorem (LDCT)). *[Roy68] Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of random variables that converge pointwise to some X (with probability 1). Suppose that there is a random variable Y such that $\langle |Y|\rangle < \infty$ and $|X_n| \le |Y|$, then*

$$\lim_{n\to\infty}\langle |X_n - X|\rangle = 0.$$

That also implies,

$$\lim_{n\to\infty}\langle X_n\rangle = \langle \lim_{n\to\infty} X_n\rangle = \langle X\rangle.$$

Definition 2.3. *Given $A \in \mathcal{F}$ We define the* ***characteristic function*** *of A, $\chi_A : \Omega \to \mathbb{R}$, by*

$$\chi_A(\omega) = \begin{cases} 1 & \text{if } \omega \in A \\ 0 & \text{if } \omega \notin A \end{cases}.$$

We will use the property $\langle \chi_A\rangle = \mathbb{P}(A)$ throughout.

Definition 2.4. *Let $h : [0,\infty) \to \mathbb{R}$ and let $a > 0$. Let*

$$\underline{m}_n(a) = \inf\{h(t) | t \in [(n-1)a, na]\}$$
$$\overline{m}_n(a) = \sup\{h(t) | t \in [(n-1)a, na]\}$$

We say h is ***directly Riemann integrable*** *if $\sum_{n=1}^{\infty} \underline{m}_n(a), \sum_{n=1}^{\infty} \overline{m}_n(a) < \infty$ and*

$$\lim_{a\to 0^+}\sum_{n=1}^{\infty} \underline{m}_n(a) = \lim_{a\to 0^+}\sum_{n=1}^{\infty} \overline{m}_n(a)$$

This is a fairly technical definition, but for our purposes we only require the following theorem.

Theorem 2.5 (Sufficiency condition for direct Riemann integrability). *[Ros96] If $h : [0,\infty) \to \mathbb{R}$ satisfies:*

(i) $h(t) \ge 0$ for all $t \ge 0$,

(ii) $h(t)$ is non-increasing,

(iii) $\int_0^{\infty} h(t)dt < \infty$,

then h is directly Riemann integrable.

2.2 Probability theory

Theorem 2.6 (Strong Law of Large Numbers (SLLN)). *Let $\{X_n\}_{n=1}^{\infty}$ be i.i.d. random variables with common distribution X, then we have that with probability 1,*

$$\lim_{n\to\infty}\frac{\sum_{i=1}^{n} X_i}{n} = \langle X\rangle.$$

Definition 2.7. *Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of independent random variables. An integer-valued random variable N is said to be a **stopping time** for $\{X_n\}_{n=1}^{\infty}$ if the the event $\{N = n\}$ is independent of $\{X_i\}_{i=n+1}^{\infty}$.*

Theorem 2.8 (Wald's Equation)**.** *[Gal13] Let N be a stopping time for a sequence $\{X_n\}_{n=1}^{\infty}$ of i.i.d. random variables with common distribution X. If $\langle X\rangle, \langle N\rangle < \infty$, then*

$$\left\langle \sum_{n=1}^{N} X_n \right\rangle = \langle N\rangle\langle X\rangle.$$

Definition 2.9. *Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of random variables with distribution functions F_n, and X be a random variable with distribution function F. We say that X_n **converges weakly** to X if*

$$\lim_{n\to\infty} F_n(x) = F(x),$$

for all continuity points x of F. We this by denoted by

$$\lim_{n\to\infty} X_n \Longrightarrow X.$$

Definition 2.10. *If X is a random variable with distribution F, then we define the **survival function** to be*

$$\overline{F} \equiv 1 - F$$

Theorem 2.11. *If X is a random variable on $\Omega = [0, \infty)$ with distribution function F, then*

$$\langle X\rangle = \int_0^{\infty} \overline{F}(x) dF(x).$$

3 The renewal process

We begin our exploration of a renewal process by first introducing a point process.

3.1 Definitions and some jargon

Definition 3.1. *A **simple point process** is a sequence of points $\{t_n\}_{n=0}^{\infty}$ such that*

$$0 \equiv t_0 < t_1 < t_2 < \dots,$$

*and $\lim_{n\to\infty} t_n = \infty$. If each t_n is a random variables then we call $\{t_n\}_{n=0}^{\infty}$ a **random point process**. We also refer to t_n as the n^{th} **arrival time** or **epoch**. Finally we call X_n the n^{th} **interarrival time**,*

$$X_n \equiv t_n - t_{n-1}, \quad \forall n \geq 1.$$

It should be noted that the "simple" in the above definition refers to the fact that only one arrival can occur at a given time. It should also be noted that by definition we have

$$t_n = \sum_{i=1}^{n} X_i. \tag{1}$$

Definition 3.2. *Let $\{t_n\}_{n=0}^{\infty}$ be a point process, we say $N(t)$ is the* ***counting process*** *for $\{t_n\}_{n=0}^{\infty}$ defined by*

$$N(t) = \begin{cases} 0 & \text{if } t = 0 \\ \max\{n | t_n \in (0,t]\} & \text{if } t > 0 \end{cases}.$$

We also define the ***renewal function*** *to be*

$$m(t) = \langle N(t) \rangle.$$

$N(t)$ counts the number of points (or the number of arrivals) that occur in the point process that lie in the interval $(0, t]$.

Definition 3.3. *Given a random point process $\{t_n\}_{n=0}^{\infty}$. If the interarrival times $\{X_n\}_{n=1}^{\infty}$ are i.i.d., then we say that the associated counting process $N(t)$ is a* ***renewal process****. Let X denote the common distribution and let $F(x) = \mathbb{P}(X \leq x)$ be the cumulative distribution of X. We refer to the t_n as the n^{th}* ***renewal time****. Finally we define the* ***rate*** *of the renewal process to be*

$$\lambda \equiv \frac{1}{\langle X \rangle}.$$

From this point onwards we will assume that we are dealing with a renewal process with notation as defined in the definition above. Our goal in the next section will be to determine properties and the limiting behaviour of N, m for a renewal process. We also note that the the function $t_{N(t)}$ will be very useful throughout. It represents the time that the last event that occurred at or before time t, and similarity $t_{N(t)+1}$ represents the time of the first event after time t. [Sig09]

Example 3.4. We have already seen in class that a Poisson process with mean μ is a renewal process. The distribution for $N(t)$ is

$$\mathbb{P}(N(t) = n) = \frac{e^{t\mu}(t\mu)^n}{n!},$$

with the interarrival times being exponentially distributed with rate $\lambda = 1/\mu$. The Poisson process is (as we will see by the end of the paper) the model for the "perfect" renewal process, in the sense that it satisfies the properties all renewal process follow at equilibrium, but for all time.

Example 3.5. Suppose you are waiting at the bus stop and the successive bus time arrivals follow a distribution F. We define a renewal to occur whenever a bus arrives. $N(t)$ counts the number of buses that arrived since time 0. We will explore the distributions of the waiting times between successive buses, and the limiting behaviour of N in the coming sections.

4 Properties of the renewal process

We begin our exploration of the renewal process by exploring the properties of $N(t)$, and $m(t)$.

4.1 Distribution of $N(t)$

First let us determine the distribution on t_n with respect to F. (1) and the fact that X_n are i.i.d. imply that the distribution for t_n is F convolved with itself n times. We will refer to the distribution of t_n as F_n. Since $N(t) \geq n$ if and only if $t_n \leq t$, we have the following relationship:

$$\begin{aligned}\mathbb{P}(N(t) = n) &= \mathbb{P}(N(t) \geq n) - \mathbb{P}(N(t) \geq n+1)\\ &= \mathbb{P}(t_n \leq t) - \mathbb{P}(t_{n+1} \leq t)\\ &= F_n(t) - F_{n+1}(t).\end{aligned}$$

The following proposition establishes a relationship between $m(t)$ and $F_n(t)$.

Proposition 4.1. *[Ros96]* $m(t) = \sum_{n=1}^{\infty} F_n(t)$

Proof. Let $A_n = \{\omega \in \Omega | t_n(\omega) \in (0, t]\}$. Note that $\mathbb{P}(A_n) = \mathbb{P}(t_n \leq t) = F_n(t)$. Then we have that $N(t) = \sum_{n=1}^{\infty} \chi_{A_n}$.

$$m(t) = \left\langle \sum_{n=1}^{\infty} \chi_{A_n} \right\rangle = \sum_{n=1}^{\infty} \langle \chi_{A_n} \rangle = \sum_{n=1}^{\infty} \mathbb{P}(A_n) = \sum_{n=1}^{\infty} F_n(t).$$

To exchange the expectation and the sum in the above computation we used the fact that the sum is increasing and thus can apply the monotone convergence theorem (MCT). □

It should also be noted that if you take Laplace transforms of both sides in 4.1 we get

$$\tilde{m} = \mathcal{L}\{m\} = \sum_{n=1}^{\infty} \mathcal{L}\{F_n\} = \sum_{n=1}^{\infty} \tilde{F}_n = \sum_{n=1}^{\infty} (\tilde{F})^n = \frac{\tilde{F}}{1 - \tilde{F}}.$$

Thus we also have

$$m = \mathcal{L}^{-1}\left\{\frac{\tilde{F}}{1 - \tilde{F}}\right\}.$$

We will also give a formula for the cumulative distribution function $F_{t_{N(t)}}$, of $t_{N(t)}$, which will come in handy when talking about reward renewal process.

Proposition 4.2. *[Ros96]* $F_{t_{N(t)}}(x) = \overline{F}(t) + \int_0^x \overline{F}(t-y) dm(y) \quad 0 \leq s \leq t.$

Proof.

$$\begin{aligned}\mathbb{P}(t_{N(t)} \leq x) &= \sum_{n=0}^{\infty} \mathbb{P}(t_n \leq x, t_{n+1} > t)\\ &= \mathbb{P}(t_1 > t) + \sum_{n=1}^{\infty} \mathbb{P}(t_n \leq x, t_{n+1} > t)\\ &= 1 - F(t) + \sum_{n=1}^{\infty} \int_0^{\infty} \mathbb{P}(t_n \leq x, t_{n+1} > t | t_n = y) dF_n(y)\\ &= \overline{F}(t) + \sum_{n=1}^{\infty} \int_0^{x} \mathbb{P}(t_{n+1} > t | t_n = y) dF_n(y)\\ &= \overline{F}(t) + \sum_{n=1}^{\infty} \int_0^{x} \mathbb{P}(t_{n+1} - t_n > t - y) dF_n(y)\\ &= \overline{F}(t) + \int_0^{x} \mathbb{P}(X_{n+1} > t - y) \sum_{n=1}^{\infty} dF_n(y)\\ &= \overline{F}(t) + \int_0^{x} \overline{F}(t-y) dm(y).\end{aligned}$$

We were able to swap the sum and the integral because the sum of was increasing and thus MCT applies. The last line used 4.1. $\square$

4.2 Limiting behaviour of $N(t)$

We now want to investigate the long term behaviour of N. We begin by first finding a linear approximation for N.

Theorem 4.3 (SLLN for Renewal Process)**.** *We have that $N(t) \sim \lambda t$ with probability 1.*

Proof. This is a consequence of the strong law of large numbers. Indeed we have

$$t_{N(t)} \leq t < t_{N(t)+1}. \tag{2}$$

Using (1) and by dividing (2) by $N(t)$ we get,

$$\frac{\sum_{n=1}^{N}(t) X_n}{N(t)} \leq \frac{t}{N(t)} < \frac{\sum_{n=1}^{N(t)+1} X_n}{N(t)}. \tag{3}$$

The left and right hand side of (3) both converge to $\langle X \rangle$ with probability 1 by the strong law of large numbers. Thus by the squeeze theorem we have

$$\lim_{t\to\infty} \frac{t}{N(t)} = \langle X \rangle,$$

or equivalently

$$\lim_{t\to\infty} \frac{\lambda t}{N(t)} = 1.$$

$\square$

One can also ask, is $m(t) \sim \lambda t$? The answer turns out to be yes and is refereed to in the literature as the elementary renewal theorem. The proof is not as simple as taking expectations of both sides in the above proposition. Since in general the expectation and limits don't commute. Before we work out the proof, we will prove the following lemma.

Lemma 4.4. *If $\lambda > 0$ then $N(t) + 1$ is a stopping time and*

$$\langle t_{N(t)+1} \rangle = \frac{m(t) + 1}{\lambda}.$$

Proof. First let us show that $N(t) + 1$ is a stopping time for $(X_n)_n$. Let $n \in \mathbb{N}$, then we have

$$\begin{aligned} N(t) + 1 = n &\Leftrightarrow N(t) = n - 1 \\ &\Leftrightarrow \sum_{i=1}^{n-1} X_i \leq t, \quad \sum_{i=1}^{n} X_i > t. \end{aligned}$$

Since the event $\{N(t) + 1 = n\}$ only depends on $X_1, \ldots, X_n$, we have $N(t) + 1$ is a stopping time.

$$\langle t_{N(t)+1} \rangle = \left\langle \sum_{n=1}^{N(t)+1} X_n \right\rangle = \langle X \rangle \langle N(t) + 1 \rangle = \frac{m(t) + 1}{\lambda}$$

The second equality is a direct consequence of (1) and Wald's equation, $\square$

Theorem 4.5 (Elementary renewal theorem)**.** *[Gal13] $m(t) \sim \lambda t$.*

Proof. We know that $t_{N(t)+1} > t$ so by taking expectation and using the lemma above we have

$$\frac{m(t)+1}{\lambda} > t,$$

which implies that

$$\liminf_{t\to\infty} \frac{m(t)}{t} = \liminf_{t\to\infty} \frac{m(t)+1}{t} \geq \lambda. \tag{4}$$

The other inequality requires a bit more work. We begin by truncating our X_n by $B > 0$.Define

$$X_{n,M} = \begin{cases} X_n & \text{if } X_n \leq B \\ M & \text{if } X_n > B \end{cases}.$$

Let $t_{n,B}, X_{n,B}, m_B(t), \lambda_B$ be the arrival times, interarrival times, and renewal function for the truncated renewal process $N_B(t)$ respectively. Note that we have the following relations between the two processes. First $t_{n,B} \leq t_n$, which implies that $N_B(t) \geq N(t)$ and $m_B(t) \geq m(t)$. Also we note that $X_{n,B} \leq X_n$ and $X_{n,B}$ converge pointwise to X_n, so by LDCT we have

$$\lim_{B\to\infty} \langle X_{n,B}\rangle = \langle X_n\rangle = \langle X\rangle,$$

or equivalently,

$$\lim_{B\to\infty} \lambda_B = \lambda.$$

Now we have that $t_{n+1,M} \leq t + M$, so again by taking expectations and using the above lemma we have

$$\frac{m_B(t)+1}{\lambda_B} \leq t + B,$$

thus implying, analogous to the above, that

$$\limsup_{t\to\infty} \frac{m(t)}{t} \leq \limsup_{t\to\infty} \frac{m_B(t)}{t} \leq \lambda_B \to \lambda, \quad B \to \infty. \tag{5}$$

(4), (5) together give us our result. □

We will see the implications of the elementary renewal theorem in the coming sections.

4.3 Central limit theorem for renewal process

We end off this section by proving an analogue of the central limit theorem to show how the distribution of $N(t)$ evolves with time. First note that by (1) and the fact that X_n are i.i.d., we have the following result as a direct consequence of the central limit theorem.

$$\lim_{n\to\infty} Z_n \equiv \frac{t_n - n/\lambda}{\sigma\sqrt{n}} \Longrightarrow Z,$$

Where Z is a normally distributed with mean 0 and variance 1, and σ is the standard deviation of X. We now wish to achieve an analogous result for $N(t)$.

Theorem 4.6 (CLT for Renewal Process)**.** *[Sig09]*

$$\lim_{t\to\infty} Z(t) \equiv \frac{N(t)-\lambda t}{\sigma\sqrt{\lambda^3 t}} \Longrightarrow Z.$$

Proof. We will use the fact that $\mathbb{P}(N(t) < n) = \mathbb{P}(t_n > t)$ which will allow us to use properties of t_n to deduce the limiting behaviour of N. Let us fix an $x \in \mathbb{R}$, and let $n(t) = \lambda t + x\sqrt{\sigma^2\lambda^3 t}$. We can assume without loss of generality that $n(t)$ is an integer, since we can always just restrict ourselves to an unbounded sequence of t_n such that $n(t)$ is an integer (by continuity of $n(t)$).

$$\begin{aligned}\mathbb{P}(Z(t) < x) &= \mathbb{P}(N(t) < n(t)) \\ &= \mathbb{P}(t_{n(t)} > t) \\ &= \mathbb{P}\left(\frac{t_{n(t)} - n(t)/\lambda}{\sigma\sqrt{n(t)}} > \frac{t - n(t)/\lambda}{\sigma\sqrt{n(t)}}\right) \\ &= \mathbb{P}\left(Z_{n(t)} > \frac{-x}{\sqrt{1 + x\sigma/\sqrt{t/\lambda}}}\right).\end{aligned}$$

The last line was just substituting in $n(t)$ and simplifying. Now $\lim_{t\to\infty} Z_{n(t)} = \lim_{n\to\infty} Z_n \Longrightarrow Z$ and $\lim_{t\to\infty}\sqrt{1 + x\sigma/\sqrt{t/\lambda}} = 1$. Thus

$$\lim_{t\to\infty} \mathbb{P}(Z(t) < x) = \mathbb{P}(Z > -x) = \mathbb{P}(Z < x)$$

The last equality used the symmetry of the normal distribution. $\square$

This theorem gives an alternate proof of 4.3, and actually generalizes it. In addition is also tells us that $\mathrm{Var}(N(t)) \sim \sigma^2\lambda^3 t$ with probability 1. All of the limiting results in this section will also remain valid for delayed renewal processes.

5 Chapter of named theorems

5.1 Key renewal theorem

[Gal13] [Ros96] We begin by defining the notion of a lattice.

Definition 5.1. *A non-negative random variable X is said to be a* ***lattice*** *if there exists $d \geq 0$ such that*

$$\sum_{n=0}^{\infty} \mathbb{P}(X = nd) = 1.$$

The largest d with this property is said to be the ***period*** *of X.*

In other words we have that a random variable is a lattice if and only if it only takes on values in some additive subgroup of the real line. The next theorem is often referred to as the renewal theorem.

Theorem 5.2 (Blackwell's theorem/Renewal theorem)**.**

(i) If X is a lattice, then for all $a \geq 0$ we have,

$$\lim_{t\to\infty} m(t+a) - m(t) = a\lambda.$$

(ii) If X is a lattice with period d, then

$$\lim_{n\to\infty} \langle \textit{number of renewals at } nd \rangle = \lambda d.$$

Remark 5.3. (i) The proof of this theorem is fairly long and technical and is thus ommitted from this exposition. For the fist part of the theorem, the difficulty comes in proving the limit actually exists. If it does exist, it is a fairly simple consequence of the elementary renewal theorem that the limit must be $a\lambda$. To see this, let us define

$$g(a) \equiv \lim_{t\to\infty} m(t+a) - m(t), \quad \forall a \geq 0.$$

First note that

$$\begin{aligned} g(a+b) &= \lim_{t\to\infty} [m(t+a+b) - m(t)] \\ &= \lim_{t\to\infty} [m(t+a+b) - m(t+b) + m(t+b) - m(t)] \\ &= g(a) + g(b). \end{aligned}$$

Thus g is an additive function and is increasing since m is increasing. It is well known (shown in an introductory analysis course) that this implies that g is linear, so there is $c \in \mathbb{R}$ such that

$$g(a) = ca.$$

To show that c is indeed λ, define $x_n = m(n) - m(n-1)$, so we have that $\lim_{n\to\infty} x_n = g(1) = c$. Since the sequence converges to c, so does the average of the terms. Thus

$$\begin{aligned} c &= \lim_{n\to\infty} \frac{\sum_{i=1}^n x_i}{n} \\ &= \lim_{n\to\infty} \frac{m(n) - m(0)}{n} \\ &= \lim_{n\to\infty} \frac{m(n)}{n} \\ &= \lambda, \end{aligned}$$

where we used the fact that $m(0) = 0$ and the elementary renewal theorem. [Ros96]

(ii) The reason the first part of Blackwell's theorem does not hold for lattices is because the length of the internal is not relevant but rather the number of occurrences of nd in in the interval. What the second part of Blackwell's theorem asserts is that

$$\lim_{n\to\infty} \mathbb{P}(\text{renewal occurs at } nd) = d\lambda.$$

We now introduce the key renewal theorem, the "integral version" of Blackwell's theorem.

Theorem 5.4 (Key Renewal Theorem)**.** *If X is not a lattice and if $h(t)$ is directly Riemann integrable, then we have*

$$\lim_{t\to\infty} h * m'(t) = \lim_{t\to\infty} \int_0^t h(t-x)dm(x) = \lambda \int_0^\infty h(t)dt.$$

Again, the proof of the key renewal theorem is omitted since it is fairly technical and lengthy. The fist point of note is that it can be shown that the first part of Blackwell's theorem and the Key renewal theorem are equivalent. The Key renewal theorem implies Blackwell's theorem by letting $h = \chi_{[0,a]}$, and computing the resulting integrals. We can also show the reverse implication by showing the result for step functions (linear combinations of characteristic functions for intervals) and using the fact that they are dense in the space of directly Riemann integrable functions [D'A]. Since they are equivalent, we often use them interchangeably. The details are omitted for brevity.

5.2 Renewal reward process

We again add a slight modification to our renewal process. Suppose you have a cab driver working a shift who drops off a new passenger at time t_n, $n \geq 1$. This forms a renewal process, with interarrival times $X_n = t_n - t_{n-1}$ with $t_0 = 0$, representing the total amount of time looking for and driving a passenger. During the n^{th} renewal the driver makes a profit R_n, which are i.i.d. random variables that depends X_n. We want to determine how much money the driver will make on average per unit time. In general given a renewal process with interarrival times $\{X_n\}_{n=1}^{\infty}$, we call a family $\{R_n\}_{n=1}^{\infty}$ of i.i.d. random variables **rewards** for the n^{th} interarrival time with common distrbution R. We say that $\{(X_n, R_n)\}_{n=1}^{\infty}$ is called a **reward renewal process** if the sequence of random vectors is i.i.d. Note that R_n can depend on X_n, and it need not be positive. We call (X_n, R_n) a **reward cycle**, and $R(t)$ the **total reward** earned at time t defined by

$$R(t) \equiv \sum_{n=1}^{N(t)} R_n.$$

Our main goal this section is to determine

$$\lim_{t\to\infty} \frac{R(t)}{t}.$$

Theorem 5.5 (Renewal Reward Theorem)**.** *[Ros96] Given a reward renewal process (X_n, R_n) with $\langle |R| \rangle < \infty$, we have*

(i) With probability 1, the rate at which the reward is earned is equal to the expected reward in a cycle divided by the expected cycle length, i.e.,

$$\lim_{t\to\infty} \frac{R(t)}{t} = \frac{\langle R \rangle}{\langle X \rangle}.$$

(ii) Furthermore,

$$\lim_{t\to\infty} \frac{\langle R(t) \rangle}{t} = \frac{\langle R \rangle}{\langle X \rangle}.$$

Proof. Just like in the proof of the elementary renewal theorem, the first statement is straightforward, but the second one takes some work.

(i)

$$\lim_{t\to\infty} \frac{R(t)}{t} = \lim_{t\to\infty} \frac{\sum_{n=1}^{N(t)} R_n}{t} = \lim_{t\to\infty} \frac{\sum_{n=1}^{N(t)} R_n}{N(t)} \frac{N(t)}{t} = \langle R \rangle \lambda = \frac{\langle R \rangle}{\langle X \rangle}.$$

The second last equality used the strong law of large numbers and the elementary renewal theorem.

Just as in the proof of the elementary renewal theorem we have that $N(t)+1$ is a stopping time for $(X_n)_n$ and for $(R_n)_n$ (the proof for the latter case is identical). So by Wald's equation we have

$$\langle R(t)\rangle = \left\langle \sum_{n=1}^{N(t)+1} R_n \right\rangle - \langle R_{N(t)+1}\rangle = (m(t)+1)\langle R\rangle - \langle R_{N(t)+1}\rangle$$

Therefore, with an application of the elementary renewal theorem, we get,

$$\begin{aligned}\lim_{t\to\infty} \frac{\langle R(t)\rangle}{t} &= \lim_{t\to\infty}\left[\frac{m(t)+1}{t}\langle R\rangle - \frac{\langle R_{N(t)+1}\rangle}{t}\right] \\ &= \frac{\langle R\rangle}{\langle X\rangle} - \lim_{t\to\infty}\frac{\langle R_{N(t)+1}\rangle}{t}.\end{aligned}$$

We are done if we can show that the last term is 0. Let $g(t) = \langle R_{N(t)+1}\rangle$. We condition g by $t_{N(t)+1}$,

$$g(t) = \langle R_{N(t)+1}|t_{N(t)} = 0\rangle \mathbb{P}(t_{N(t)} = 0) + \int_0^t \langle R_{N(t)+1}|t_{N(t)} = s\rangle dF_{t_{N(t)}}.$$

First note that

$$\mathbb{P}(t_{N(t)} = 0) = \mathbb{P}(X_1 > t) = \overline{F}(t).$$

Also by 4.2 we get,

$$dF_{t_{N(t)}} = \overline{F}(t-s)dm(s).$$

Finally,

$$\begin{aligned}\langle R_{N(t)+1}|t_{N(t)} = 0\rangle &= \langle R_1|X_1 > t\rangle \\ \langle R_{N(t)+1}|t_{N(t)} = s\rangle &= \langle R|X > t-s\rangle.\end{aligned}$$

Therefore,

$$g(t) = \langle R_1|X_1 > t\rangle\overline{F}(t) + \int_0^t \langle R|X > t-s\rangle\overline{F}(t-s)dm(s).$$

Since

$$h(t) \equiv \langle R|X > t\rangle\overline{F}(t) = \int_t^\infty \langle R|X = x\rangle dF(x) \le \int_0^\infty \langle |R||X = x\rangle dF(x) = \langle |R|\rangle =< \infty,$$

we have,

$$\lim_{t\to\infty} h(t) = 0$$

Let $\varepsilon > 0$ We can thus find a $T > 0$ such that $|h(t)| < \varepsilon\langle X\rangle$ for all $t > T$. Thus we have

$$\begin{aligned}\lim_{t\to\infty}\frac{g(t)}{t} &= \lim_{t\to\infty}\left|h(t) + \int_0^t h(t-s)dm(s)\right| \\ &\le \lim_{t\to\infty}\frac{|h(t)|}{t} + \int_0^{t-T}\frac{|h(t-s)|}{t}dm(s) + \int_{t-T}^t \frac{|h(t-s)|}{t}dm(s) \\ &\le \lim_{t\to\infty}\frac{\varepsilon\langle X\rangle}{t} + \varepsilon\langle X\rangle\frac{m(t-T)-m(0)}{t} + \langle |R|\rangle\frac{m(t)-m(t-T)}{t} \\ &= 0 + \lambda\epsilon\langle X\rangle + \langle |R|\rangle(\lambda-\lambda) \\ &= \varepsilon\end{aligned}$$

We used the the triangle inequality and the elementary renewal theorem. □

Example 5.6. Note that this example is slight a modification of one in [Ros96]. Suppose we have a train that leaves only when N_0 passengers arrive. Further suppose that the cost associated with waiting is nc dollars per unit time, where n is the number of passengers that are waiting and c is a constant. We want to determine the average cost incurred before the train leaves. We define a renewal to occur whenever the train leaves. Let X_n be the interarrival times of the passengers in a given cycle, we assume there are i.i.d, with common distribution. We have that the train leaves at time $\sum_{n=1}^{N_0} X_n$, so the average length of the cycle is

$$N_0\langle X\rangle.$$

So we have the average cost of the cycle is,

$$\langle \text{cost of cycle}\rangle = \langle cX_1 + 2cX_2 + \cdots + (N_0-1)cX_{N_0-1}\rangle = \frac{2\langle X\rangle N_0(N_0-1)}{2}.$$

So by the reward renewal theorem, the average cost is

$$\frac{c(N_0-1)}{2},$$

and is independent of the how often the passengers arrive!

5.3 Alternating renewal processes

Before we can begin talking about waiting times, let us make a digression. Consider a system where there are two states, on and off, and the system alternates between the two, like a light switch. Suppose the system is on initially for a (non-negative) time Y_1, then off for a (non-negative) time Z_1, followed by being on for a time Y_2, etc.

We call the sequence of random vectors $\{(Y_n, Z_n)\}_{n=1}^{\infty}$ an **alternating renewal process** if they are i.i.d.. Notice that the counting function associated with the interarrival times $\{Y_n\}_{n=1}^{\infty}$, $\{Z_n\}_{n=1}^{\infty}$ and, $\{X_n\}_{n=1}^{\infty}$, where $X_n = Z_n + Y_n$, are renewal processes. Note we do not require Y_n and Z_n to be independent, but we do have Y_n, Z_{n-1} are independent. (Z_n, Y_n) is called a **on-off cycle**.

We will let F, G, H denote the CDF's for X, Y, Z, the common distributions for X_n, Y_n, Z_n respectively. We also define

$$\begin{aligned} P(t) &\equiv \mathbb{P}(\text{system is on at time t}) \\ Q(t) &\equiv \mathbb{P}(\text{system is off at time t}) = 1 - P(t). \end{aligned}$$

Theorem 5.7 (Alternating Renewal Process Theorem)**.** *[Sig09] Given an alternating renewal process $\{(Z_n, Y_n)\}_{n=1}^{\infty}$, where X is not lattice, and $\langle X\rangle < \infty$, then:*

$$\begin{aligned} \lim_{t\to\infty} P(t) &= \frac{\langle Y\rangle}{\langle Z\rangle + \langle Y\rangle} = \frac{\langle Y\rangle}{\langle X\rangle}, \\ \lim_{t\to\infty} Q(t) &= \frac{\langle Z\rangle}{\langle Z\rangle + \langle Y\rangle} = \frac{\langle Z\rangle}{\langle X\rangle}. \end{aligned}$$

Proof. Given an alternating renewal process, suppose we earn at a rate of one per unit time whenever the system is "on", and do not earn anything when the system is "off". Then the reward R_n in a given on-off cycle is given by Y_n. We also have that the total reward $R(t)$ in the total time the system is on in $[0, t]$. So

$$\lim_{t\to\infty} P(t) = \lim_{t\to\infty} \frac{\text{total time system is ``on'' in } [0,t]}{t} = \lim_{t\to\infty} \frac{R(t)}{t} = \frac{\langle R\rangle}{\langle X\rangle} = \frac{\langle Y\rangle}{\langle X\rangle}.$$

The second last equality used the renewal reward theorem. □

We will see the implications of these results in the next section when we consider waiting times.

6 Waiting for the bus to come

In this section we will examine the behaviour of the interarrival times. In particular assuming we are at a time t in the interval, we want to determine how long it has been since the last arrival and how much time is remaining until the next arrival. With the previous section, we now have enough machinery to explore this topic.

Definition 6.1. *We define $A(t)$ to be the* ***age*** *at t, and $B(t)$ to be the* ***excess*** *or* ***residual life*** *at t by the following formulas.*

$$\begin{aligned} A(y) &\equiv t - t_{N(t)}, \\ B(t) &\equiv t_{N(t)+1} - t. \end{aligned}$$

We also define the ***spread*** *as the length of the interarrival times given by*

$$S(t) = t_{N(t)+1} - t_{N(t)} = A(t) + B(t).$$

$A(t)$ denotes the time since the last renewal, and similarly $B(t)$ denotes the time from t until the next renewal event occurs. For example, suppose you are at a bus stop, we say a renewal event occurs every time a bus arrives. In that scenario $A(t)$ represents the time by which you missed the previous bus by, $B(t)$ represents the time left until the next bus arrives, and $S(t)$ is the time between the bus you missed and the bus you will catch. For the remainder of the section we will assume that the X is not lattice.

6.1 Age, excess, spread OH MY!

We want to determine the properties of A, B, S. Let us begin by determining their distributions, at least in the limit. Let us begin by fixing an x, and define the on-off cycle by saying the system is "on" if $A(t) \leq x$, and "off" otherwise. With Y_n, Z_n as defined in the previous section we have, $Y = \min(X, x)$, $Y + Z = X$, Then by 5.7 we have

$$\begin{aligned} \lim_{t\to\infty} \mathbb{P}(A(t) \leq x) &= \lim_{t\to\infty} \mathbb{P}(\text{system is "on" at time } t) \\ &= \frac{\langle \min(X, x) \rangle}{\langle X \rangle} \\ &= \lambda \int_0^\infty \mathbb{P}(\min(X, x) > y) dy \\ &= \lambda \int_0^x \overline{F}(y) dy \end{aligned}$$

Now to determine the distribution of $B(t)$, let use the the same on-off cycle from the above, So the off time in the cycle is $\min(x, X)$, so as before

$$\begin{aligned} \lim_{t\to\infty} \mathbb{P}(B(t) \leq x) &= \lim_{t\to\infty} \mathbb{P}(\text{system is "off" at time} t) \\ &= \frac{\langle \min(X, x) \rangle}{\langle X \rangle} \\ &= \lim_{t\to\infty} P(A(t) \leq x) \end{aligned}$$

Thus we have shown that both $A(t)$ and $B(t)$ have the same limiting distribution. (give some intuition).

Finally we want to determine the distribution of the spread. Again we will construct an alternating renewal process and apply 5.7. We define the system to be "on" if the renewal interval is greater than x and is "off" otherwise. So we have

$$\begin{aligned}\mathbb{P}(S(t) > x) &= \mathbb{P}(\text{length of renewal internval containing } t > x) \\ &= \mathbb{P}(\text{system is on at time } t).\end{aligned}$$

Thus by 5.7 we have

$$\begin{aligned}\lim_{t\to\infty} \mathbb{P}(S(t) > x) &= \frac{\langle \text{on time in cycle} \rangle}{\langle X \rangle} \\ &= \lambda \langle X | X > x \rangle \mathbb{P}(X > x) \\ &= \lambda \int_0^\infty \frac{y\chi_{(x,\infty)}}{\mathbb{P}(X > x)} dF(y) \mathbb{P}(X > x) \\ &= \lambda \int_x^\infty y dF(y).\end{aligned}$$

Thus we have

$$\lim_{t\to\infty} \mathbb{P}(S(t) \le x) = \lambda \int_0^x y dF(y).$$

[Sig09]

Our next goal is to determine the average age, excess and spread of the renewal process for large time, i.e. to determine relationships for the following:

$$\lim_{t\to\infty} \frac{\int_0^t A(s)ds}{t}, \quad \lim_{t\to\infty} \frac{\int_0^t B(s)ds}{t}, \quad \lim_{t\to\infty} \frac{\int_0^t S(s)ds}{t}$$

Theorem 6.2. *[Ros96] [Sig09]*

(i) With probability 1,

$$\lim_{t\to\infty} \frac{\int_0^t A(s)ds}{t} = \lim_{t\to\infty} \frac{\int_0^t B(s)ds}{t} = \frac{\langle X^2 \rangle}{2\langle X \rangle}, \quad \lim_{t\to\infty} \frac{\int_0^t S(s)ds}{t} = \frac{\langle X^2 \rangle}{\langle X \rangle}$$

(ii)

$$\lim_{t\to\infty} \frac{\int_0^t \langle A \rangle(s)ds}{t} = \lim_{t\to\infty} \frac{\int_0^t \langle B \rangle(s)ds}{t} = \frac{\langle X^2 \rangle}{2\langle X \rangle}, \quad \lim_{t\to\infty} \frac{\int_0^t \langle S \rangle(s)ds}{t} = \frac{\langle X^2 \rangle}{\langle X \rangle}$$

Proof. (i) Let us first show the result for $A(t)$. Suppose we are being paid a rate equal to age of the renewal process at that time. So for time $s < t$ we are being paid a rate $A(s)$ and our total earnings are

$$\int_0^t A(s)ds.$$

In a given cycle X, the age of the renewal process at time s is just s, so we have the total earning in a cycle is

$$\int_0^X sds = \frac{X^2}{2}.$$

Thus we have by the reward renewal theorem

$$\lim_{t\to\infty} \frac{\int_0^t A(x)dx}{t} = \frac{\langle X^2\rangle}{2\langle X\rangle}.$$

For $B(t)$, suppose we are being paid a rate equal to the excess of the renewal process at that time. So for time $s < t$ we are being paid a rate $B(s)$ and our total earnings are

$$\int_0^t B(s)ds.$$

In a given cycle X, the age of the renewal process at time s is just $X - s$, so the total earning in a cycle is

$$\int_0^X X - sds = \frac{X^2}{2}.$$

Thus by the renewal reward theorem,

$$\lim_{t\to\infty} \frac{\int_0^t B(x)dx}{t} = \frac{\langle X^2\rangle}{2\langle X\rangle}.$$

Finally,

$$\lim_{t\to\infty} \frac{\int_0^t S(x)dx}{t} = \lim_{t\to\infty} \frac{\int_0^t A(x)dx}{t} + \lim_{t\to\infty} \frac{\int_0^t B(x)dx}{t} = \frac{\langle X^2\rangle}{2\langle X\rangle} + \frac{\langle X^2\rangle}{2\langle X\rangle} = \frac{\langle X^2\rangle}{\langle X\rangle}.$$

(ii) This proof is nearly identical to the proof of the first part of the theorem, but using the second part of the renewal reward theorem instead.

□

Since $\mathrm{Var}(X) = \langle X^2\rangle - \langle X\rangle^2 \geq 0$, we have that

$$\lim_{t\to\infty} \frac{\int_0^t S(x)dx}{t} = \frac{\langle X^2\rangle}{\langle X\rangle} \geq \langle X\rangle$$

with equality if and only if $\mathrm{Var}(X) = 0$ if and only if X is constant. So if X is not constant, then in the limit the average spread in strictly bigger then the average length of the interarrival time! We will show that this is a special case of a bizarre phenomenon called the inspection paradox.

6.2 Inspection Paradox

We now prove a very non-intuitive result referred to in the literature as "the inspection paradox". It essentially states that if you wait an arbitrary time t, then the interval containing t is "in general" larger than the time it takes to observe the successive event. The average spread is greater than or equal to the average length of the interarrival time for all time! This is made rigorous below.

Theorem 6.3 ("The inspection paradox")**.** *[Sig09] For every fixed $t \geq 0$, $S(t)$ is stochastically larger than X, that is*

$$\mathbb{P}(S(t) > x) \geq \mathbb{P}(X > x) = \overline{F}(x), \quad \forall x \geq 0.$$

Moreover,

$$\langle S(t)\rangle \geq \langle X\rangle.$$

Proof.

$$\begin{aligned}
\mathbb{P}(S(t) > x) &= \langle \mathbb{P}(S(t) > x | t_{N(t)}) \rangle \\
&= \int_0^\infty \mathbb{P}(S(t) > x | t_{N(t)} = s) dF(s) \\
&= \int_0^\infty \mathbb{P}(S(t) > x | S(t) > t - s) dF(s) \\
&= \int_0^\infty \frac{\mathbb{P}(S(t) > x, S(t) > t - s)}{\mathbb{P}(S(t) > t - s)} dF(s) \\
&= \int_0^\infty \frac{\overline{F}(\max(x, t - s))}{\overline{F}(t - s)} dF(s).
\end{aligned}$$

If $x > t - s$ then,

$$\frac{\overline{F}(\max(x, t - s))}{\overline{F}(t - s)} = \frac{\overline{F}(x)}{\overline{F}(t - s)} \geq \overline{F}(x).$$

Since $\overline{F} \leq 1$. if $x \leq t - s$ then,

$$\frac{\overline{F}(\max(x, t - s))}{\overline{F}(t - s)} = \frac{\overline{F}(t - s)}{\overline{F}(t - s)} = 1 \geq \overline{F}(x).$$

Either way we have

$$\mathbb{P}(S(t) > x) \geq \int_0^\infty \overline{F}(x) dF(s) = \overline{F}(x) = \mathbb{P}(X > x)$$

To get the other part of the result we take expectations of the above line.

$$\langle S(t) \rangle = \int_0^\infty P(S(t) > x) dF(x) \geq \int_0^\infty P(X > x) dF(x) = \langle X \rangle$$

□

Let us build some intuition for this with a rather extreme example

Example 6.4. Consider lightbulbs that have a lifetime of 1 with probability $p > 0$, and they are defective (blow out immediately) with probability $1 - p$. If you observe a burning light bulb, it will always be a working one, and thus you will never see a defective one. Thus your expected spread is always 1, but the average lifetime is p. Indeed, if X is the lifespan of the bulb,

$$\frac{\langle X^2 \rangle}{\langle X \rangle} = \frac{0(1 - p) + 1^2 p}{0(1 - p) + 1p} = 1 > p = 0(1 - p) + 1p = \langle X \rangle$$

7 Equilibrium Distribution of a Renewal Process

The goal of this section is to determine when a renewal process has the properties of it's limiting distribution for all time. Before we can answer this question we need to take a diversion into delayed renewal processes.

7.1 Delayed Renewal Process

Often times were are interested in working with a process such that the first interarrival time has a different distribution then the other inter arrival times. In this case the renewal process is initiated after the first occurrence of the event. This is made rigorous in the following definition.

Definition 7.1. *Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of independent non-negative random variables. Suppose that X_1 has distribution G and $\{X_n\}_{n=2}^{\infty}$ are i.i.d. with a common distribution F. Again let $t_0 = 0$ and $t_n = \sum_{i=1}^{n} X_i$ for $n \geq 1$. We say that $N_D(t)$ is a* ***delayed renewal process****, where*

$$N_D(t) = \sup\{n | t_n \leq t\}.$$

We also define $m_D(t) = \langle N_D(t) \rangle$ to be the ***delayed renewal function****. The* ***associated renewal process*** *of the delayed renewal sequence is the renewal process made by $\{X_n\}_{n=2}^{\infty}$. The* ***rate*** *of the delayed renewal sequence is defined to be the rate of the associated renewal sequence, and is also denoted as λ.*

A delayed renewal process is clearly a special case of the renewal process when $G = F$. Analogous to the renewal case, for a delayed renewal process we have

$$\mathbb{P}(\mathbb{N}_D(t) = n) = \mathbb{P}(t_n \leq t) - \mathbb{P}(t_{n+1} \leq t) = G * F_{n-1}(t) - G * F_n(t),$$

and,

$$m_D(t) = \sum_{n=1}^{\infty} G * F_{n-1}(t).$$

Taking Laplace transforms of both sides we get

$$\tilde{m}_D(s) = \frac{\tilde{G}(s}{1 - \tilde{F}(s)}. \tag{6}$$

Finally we also have analogous limit theorems for the delayed renewal process, with identical proofs, as summarized in the following theorem.

Theorem 7.2 (Limiting Behaviour for Delayed Renewal Process)**.** *[Ros96] Given a delayed renewal process $N_D(t)$ the following hold.*

(i) With probability 1

$$\lim_{t \to \infty} \frac{N_D(t)}{t} = \lambda$$

(ii)

$$\lim_{t \to \infty} \frac{m_D(t)}{t} = \lambda$$

(iii) If F is not lattice, then $\forall a \geq 0$

$$\lim_{t \to \infty} m_D(t + a) - m_D(t) = a\lambda$$

(iv) If F, G are lattice with period d, then

$$\lim_{n \to \infty} \langle \textit{number of renewals at } nd \rangle = d\lambda$$

(v) If F is not lattice, $\langle X \rangle < \infty$, and h is directly Riemann integrable, then

$$\lim_{t\to\infty} \int_0^\infty h(t-x)dm_D(x) = \lambda \int_0^\infty h(t)dt$$

These results are not surprising since we are interested in the limiting behaviour of these processes. The delayed renewal process only differs from the renewal process by the first term, thus it makes sense that in the limit, they both produce the same outcomes.

7.2 Equilibrium renewal process

Note that the results in section except the last example are taken from [Ros96] but with more fleshed out proofs. Analogous to how we proved 4.2 for the ordinary renewal case, we can show with nearly identical proof that the distribution for $t_{N(t)}$ is given by

$$\mathbb{P}(t_{N(t)} \leq x) = \overline{G}(t) + \int_0^x \overline{F}(t-y)dm_D(y). \tag{7}$$

Definition 7.3. *Suppose $\lambda \neq 0$, then we define the **equilibrium distribution** for a renewal process to be*

$$F_e(x) \equiv \lambda \int_0^x \overline{F}(y)dy, \quad x \geq 0$$

*A delayed renewal process is called an **equilibrium renewal process** if $G = F_e$.*

Let us now determine the Fourier transform of the equilibrium distribution.

Lemma 7.4. $\tilde{F}_e = \mathcal{L}\{F_e\} = (\omega) = \lambda \frac{1-\tilde{F}(\omega)}{s}$.

Proof.

$$\begin{aligned}
\tilde{F}_e(\omega) &= \int_0^\infty e^{-sx} dF_e(x) \\
&= \lambda \int_0^\infty e^{-sx} \int_x^\infty dF(y)dx \\
&= \lambda \int_0^\infty \int_0^y e^{-sx} dx dF(y) \\
&= \frac{\lambda}{s} \int_0^\infty 1 - e^{-sy} dF(y) \\
&= \lambda \frac{1-\tilde{F}(s)}{s},
\end{aligned}$$

where we get from line two to three we swapped the order of integration using Fubini's theorem. ☐

Theorem 7.5. *For a delayed renewal process the following hold:*

1. $m_D(t) = \lambda t, \quad \forall t \geq 0$
2. $\mathbb{P}(B_D(t) \leq x) = F_e(x), \quad \forall t \geq 0$

Proof. 1. The first is a direct corollary to (6) and 7.4. Substituting in $\tilde{F}_e$ for $\tilde{G}$ we get

$$\tilde{m}_D(s) = \frac{\lambda}{s} = \mathcal{L}\{\lambda t\}$$

By uniqueness of Laplace transforms we get $m_D(s) = \lambda t$.

2. We begin by conditioning on $t_{N(t)}$, and using (7) to get,

$$\mathbb{P}(B_D(t) > x) = \mathbb{P}(B_D(t) > x | t_{N(t)} = 0)\overline{G}(t) + \int_0^t \mathbb{P}(B_D(t) > x | t_{N(t)} = s)\overline{F}(t-s) dm_D(s).$$

We have

$$\mathbb{P}(B_D(t) > x | t_{N(t)} = 0) = \mathbb{P}(X_1 > t + x | X_1 > t) = \frac{\overline{G}(t+x)}{\overline{G}(t)},$$
$$\mathbb{P}(B_D(t) > x | t_{N(t)} = s) = \mathbb{P}(X > t + x - s | X_1 > t - s) = \frac{\overline{F}(t+x-s)}{\overline{S}(t-s)}.$$

The first part of the theorem also tels us that $dm_D(s) = \lambda ds$. Substituting in the above equations, and using the fact that $G = F_e$, results in

$$\begin{aligned}\mathbb{P}(B_D(t) > x) &= \overline{G}(t+x) + \int_0^t \overline{F}(t+x-s) dm_D(s)\\ &= \overline{F}_e(t+x) + \lambda \int_x^{t+x} \overline{F}(t+x-s) ds\\ &= \overline{F}_e(x).\end{aligned}$$

□

The first part of the theorem shows that for an equilibrium renewal process Blackwell's theorem holds with equality for all time. We know in the limit at t goes to infinity part two holds in general, but in the case of a equilibrium renewal process we get it hold for all time.

Example 7.6. A valid question to ask is when is the equilibrium renewal process, equal to the renewal process, i.e. when is $F = F_e$?

$$F(x) = F_e(x) = \lambda \int_0^x 1 - F(t) dt \Longrightarrow F'(x) = \lambda(1 - F(x))$$

Solving this DE and using the normalizing condition we see that the only solution is $F(x) = 1 - e^{-\lambda t}$, which is the distribution associated with the Poisson renewal process. Thus for the Poisson process, is always in equilibrium, and it is the only such renewal process with that property. Thus in a sense the Poisson process is the "perfect" case of a renewal process.

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