Mathematical methods to capture shape

Saifuddin Syed, #20387835
University of Waterloo
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Abstract

In this paper we will explore the metric space structure on Riemannian manifolds induced by the metric. This will allow us to use analytic methods to find singularities in our manifold. We will then switch focus to curvature by introducing sectional, Ricci, and Weyl curvature, with their geometric interpretations. Finally we will view the non-triviality of the parallel transport from the algebraic setting by studying the holonomy group of a connection. We will show how the Riemannian holonomy related to the curvature tensor, and how it can be used to determine local properties of a manifold.

1 Introduction

It should be noted that we utilized [Kem13] throughout will only be cited here.

A **smooth manifold** M is a Hausdorff, second countable topological space with a smooth structure, with the property that every point contains a neighbourhood that is diffeomorphic to $\mathbb{R}^n$. On an arbitrary manifold there is no concept length, distance, or curvature thus making it very difficult to talk about their shape.

Definition. Let M be a manifold, then a **pseudo-Riemannian metric** g is a symmetric, non-degenerate $(0, 2)$-tensor on M. If g is pseudo-Riemannian metric, which is positive definite in the sense that $g(X, X) \geq 0$ for all vector fields X, then g is called a **Riemannian metric**. The pair (M, g) is called a (pseudo-)Riemannian manifold.

A (pseudo-)Riemannian metric, as we will see allows us to a notion distance, length, and curvature. With this added structure we can impose classical concepts from geometry to the abstract manifold setting.

2 Manifold as a metric space

Before we can make (M, g) into a metric space, we first need a notion of length of a curve.

Definition. Let $\gamma : [a, b] \to M$ be a segment of some smooth curve. We define the $L(\gamma)$ to be the **length** of γ, given by

$$L(\gamma) = \int_a^b |\gamma'(t)|_{g(\gamma(t))} dt,$$

where $|X|_{g(p)} \equiv \sqrt{g_p(X, X)}$.

One can show that L is independent of reparametrization of γ and is clearly non-negative. Before we define a distance function, we need to extend out definition of length to broader class of curves.

Definition. Let $\gamma : [a, b] \to M$ be a continuous segment.

- We say γ is **regular** if γ is smooth and $\gamma'(t) \neq 0$ for all $t \in [a, b]$.
- We say γ is **piecewise regular curve segment** or an **admissible curve** if there is a subdivision $a = a_0 < a_1 < \cdots < a_k = b$ such that each $\gamma|_{[a_{i-1},a_i]}$ is a regular.
- We also consider the constant curve as $\gamma : \{a\} \to M$, $\gamma(a) = p$ to be admissible.

We can now extend our definition of length to an admissible curve by summing up the lengths of each smooth segment. It can be shown that it is too independent of reparametrization. We now have built up the machinery to define the notion of distance on a manifold.

Definition. Let $p, q \in M$. The **distance** $d(p, q)$ **between** p **and** q is defined as

$$d(p, q) \equiv \inf\{L(\gamma)|\gamma \text{ is a curve from } p \text{ to } q\}$$

Theorem. d is a well-defined distance function in the sense that it satisfies the following identities:

- $d(p, q) = d(q, p)$
- $d(p, q) \leq d(p, r) + d(r, q)$
- $d(p, q) \geq 0$ where equality happens if and only if $p = q$

More over the metric space topology induced by (M, d) is the same as the underlying manifold topology.

2.1 Length minimization

Definition. An admissible curve γ on M is said to be **minimizing** if $L(\gamma) \leq L(\tilde{\gamma})$ for all admissible curves $\tilde{\gamma}$ with the same end points at γ.

Clearly a curve is minimizing if and only if $L(\gamma) = d(\gamma(a), \gamma(b))$. In general a Riemannian manifold need not have any minimizing curves between two points, for example if you look at $\mathbb{R}^2 \backslash \{(0, 0)\}$ with the euclidean metric, there is no minimizing curve between the points $(-1, 0)$ and $(1, 0)$. Also even if such a curve exists, it need not be unique. For example, any two antipodal points on the 2-sphere have infinitely many minimizing curves. So in thinking of L as a functional that takes admissible curves and outputs their length, we want to find the admissible curves that minimize L.

Theorem. [Lee97] The unit speed admissible curves γ are the critical points of L if and only if γ is a geodesic. In particular, every minimizing curve is a geodesic.

Theorem. [Lee97] Every geodesic $\gamma : I \to M$ is locally length minimizing in the sense that for every $t \in I$, there is an open neighbourhood $U \subset I$ such that $\gamma|_U$ is length minimizing between any two points on $\gamma|_U$.

This implies that geodesics are precisely the curves that minimize the distance between points, if such curves were to exist.

2.2 Hofp-Rinow, and criterion for singularities

Recall that a manifold possesses a singularity if there is a maximal geodesic that is not defined for all time. We want find a necessary and sufficient condition to determine which Riemannian manifolds contain a singularity.

Definition. We say that a manifold is **geodesically complete** if there are no singularities in the manifold, or equivalently if every maximal geodesic is defined for all time.

Definition. A metric space (X, d) is said to be **complete** if every every Cauchy sequence converges.

It turns out that these two notions of completeness are linked by the following theorem.

Theorem (Hofp-Rinow)**.** [Lee97] A connected Riemannian manifold (M, g) is geodesically complete if and only if (M, d) is complete as a metric space.

Due to this equivalence we will use complete and geodesically complete interchangeably. This is a very powerful theorem in that is gives us a way to find singularities without explicitly finding geodesics, but rather by measuring distance between points, and using the properties of metric spaces.

Corollary. If M is compact, then M has no singularities there are no singularities.

This follows from the fact that a compact metric space is complete.

Corollary. [Lee97] If there exists a point $p \in M$ such that $\exp_p$ is defined for all of T_M, then M is complete.

Theorem. [KN96] Every homogeneous Riemannian manifold is complete.

Hofp-Rinow gives us a fresh way at tackling the problem of finding singularities, because we have a completely different new set of tools from metric space theory to determine completeness as opposed to the manifold structure. This is not to say it makes the problem easier, but rather different. If fact this method is slightly limited in approach to general relativity since we can only only apply Hofp-Rinow to Riemannian manifolds, and it may not hold for general spacetime.

3 Sectional curvature

Note that this section utilized both [Lee97] [dC92] We begin by doing a quick recap of the Riemannian curvature tensor.

Definition. Recall that if ∇ is an connection on a TM then we define the **curvature tensor** R_∇ as a $(1, 3)$-tensor on M given by

$$R_\nabla(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

When ∇ is the Levi-Civita connection, we call R_∇ the Riemann curvature tensor and denote it as R.

It is often more convenient to work with a (0,4) tensor over of a (1,3) one. We do this by contracting with g.

$$R_\nabla(X, Y, Z, W) \equiv g(R_\nabla(X, Y)Z, W)$$

Theorem (Symmetries of R_∇)**.** Let ∇ be a connection on M and X, Y, Z, W be vector fields, then:

1. $R_\nabla(X, Y, Z, W) = -R_\nabla(Y, X, Z, W)$.
2. $R_\nabla(X, Y, Z, W) = -R_\nabla(X, Y, W, Z)$ if ∇ is metric compatible.
3. **(First Bianci Identity)**: $R_\nabla(X, Y, Z, W) + R_\nabla(Y, Z, X, W) + R_\nabla(Z, X, Y, W) = 0$ if ∇ is torsion-free.
4. $R(X, Y, Z, W) = R(Z, W, X, Y)$; this is only true for the Levi-Civita connection.

From this point forward we will assume that ∇ is the Levi-Civita connection.

Definition. Suppose that $n = \dim(M) \geq 2$. Let $p \in M$, Π be a 2-dimensional subspace of T_pM. Let $\{X, Y\}$ be any basis for Π. Then we define the sectional curvature of (M, g) at Π by

$$K(\Pi) \equiv \frac{R(X, Y, Y, X)}{g(X, X)g(Y, Y) - g(X, Y)^2}$$

One can show that K is actually independent of basis, and is therefore well defined. Clearly K is determined by the curvature tensor. What is surprising is that if we know the $K(\Pi)$ for all 2-planes Π in T_pM then we can reconstruct R_p. So in fact K and R encode two equivalent pieces of information. We know from lectures that R measures the non-triviality of the parallel transport along an infinitesimal parallelogram. It is a natural question to ask where the sectional curvature comes from, and why is it an intuitive way to think about curvature.

3.1 Second fundamental form

The goal of this section is to introduce sectional curvature and give a geometric interpretation. Suppose the (M, g) is a n-dimensional manifold, and let $p \in M$. There exists a open neighbourhood U of p, such that $\iota : (U, g) \to (\mathbb{R}^{n+1}, \tilde{g})$ is an injective isometric immersion, where $\tilde{g}$ is the euclidean metric and $g = \iota^*\tilde{g}$. Since we will only be doing local computations, we will assume WLOG that $M = U$. We will identify M with a surface $\iota(M)$ in $\mathbb{R}^{n+1}$ and g with $\tilde{g}$. We have that $T_p\mathbb{R}^{n+1} \cong T_pM \bigoplus N_pM$, where $N_pM \equiv (T_pM)^\perp$ is the orthogonal complement of T_pM which is identified as a subspace of $T_p\mathbb{R}^{n+1}$ since ι is an immersion.

Definition. We define the **normal bundle** of M by

$$NM \equiv \bigsqcup_{p \in M} N_pM.$$

Note that in our case NM is a 1-dimension manifold, and $T\mathbb{R}^{n+1} = TM \bigoplus NM$. There exists an orthonormal frame $\{E_1, \ldots, E_n, N\}$ for $T\mathbb{R}^{n+1}$ such that $\{E_1, \ldots, E_n\}$ is a orthonormal frame for TM and N_p is a basis for N_pM at every $p \in M$.

Theorem (The Gauss Formula)**.** Suppose that $\tilde{\nabla}$, and ∇ are the Levi-Civita connections on $\mathbb{R}^{n+1}$ and M respectively. If X, and Y are vector fields on M and $\tilde{X}, \tilde{Y}$ are any extensions to $\mathbb{R}^{n+1}$, then we have

$$\tilde{\nabla}_{\tilde{X}}\tilde{Y} = \nabla_X Y + \Pi(X, Y)N,$$

where Π is a symmetric (0,2)-tensor on M.

Π is referred to as the second fundamental form in literature:, it should be noted that it is uniquely determined up to a sign of N, so it is independent of choice. The reason for the peculiar name is because the first fundamental form is usually referred to as the metric in classical geometry of curves and surfaces. By raising one of the indices of Π we can define a (1,1)-tensor S called the shape operator defined by

$$g(X, S(Y)) = \Pi(X, Y), \quad \forall X, Y \in \Gamma(TM).$$

Since Π is symmetric, we have that $g(S(X), Y) = g(X, S(Y))$, which implies that S is selfadjoint endomorphism on TM.

Theorem (Weingarten Equation)**.** Let X be a vector field on M and let $\tilde{X}$ be any extension to $\mathbb{R}^{n+1}$. Then we have the following:

$$S(X) = -\tilde{\nabla}_{\tilde{X}} N.$$

3.2 Gauss curvature

Definition. Let $\gamma : I \to \mathbb{R}^n$ be a curve parametrized by arclength (unit speed). Then we define the **curvature of** γ **at** $t \in I$ to be

$$\kappa(t) \equiv |\ddot{\gamma}(t)|.$$

This notion of curvature of a curve comes from the classical sense, where the higher the acceleration of the curve, the more "curved" it is. We now bring our attention to geodesics. Suppose that γ is a unit speed geodesic on M. Then the Gauss equation tells us that,

$$\ddot{\gamma} = \tilde{D}_t \dot{\gamma} = D_t \dot{\gamma} + \Pi(\dot{\gamma}, \dot{\gamma}) N = \Pi(\dot{\gamma}, \dot{\gamma}) N.$$

The second equality is true since γ is a geodesic implies $D_t \dot{\gamma} = 0$. So we have $\kappa = \pm \Pi(\dot{\gamma}, \dot{\gamma})$. The geometric interpretation of Π is that for any unit vector $V \in T_pM$, $\Pi(V, V)$ is the curvature at p of the geodesic γ which starts at p, with initial velocity V. It has a positive sign if γ is curving towards N and a negative sign if it is curving away from N. Since S is self-adjoint we have that it is diagonalizable. Therefore exists an orthonormal basis $\{E_1, \ldots, E_n\}$ of eigenvectors of S with eigenvalues κ_i. Note that κ_i are uniquely determined unto sign, because Π is unique upto a sign.

Definition. The eigenvalues κ_i of S are called the **principal curvatures**, and the corresponding eigenvectors are called the **principal directions**. We define the **Gauss curvature** K by

$$K \equiv \det(S) = \prod_{i=1}^{n} \kappa_i.$$

We also define the **mean curvature** H by

$$H \equiv \frac{1}{n} \operatorname{tr}(S) = \frac{1}{n} \sum_{i=1}^{n} \kappa_i.$$

All of these concepts are unique (upto a sign) because determinant and trace are independent of basis. Since Π is a bilinear form, and curvatures of the κ_i correspond to how curvature of the geodesics which start at p and have initial velocity E_i.

3.3 Geometric interpretation

We now look at the case of $n = 2$, when we have a surface in $\mathbb{R}^3$. There is very little reason to believe that $K = \kappa_1\kappa_2$ has much to do with the intrinsic geometry of M, since it depends on the embedding. Gauss's Theorema Egregium (roughly translates "totally awesome theorem"), which is considered one of the most remarkable discoveries in classical geometry, says that despite all odds K is invariant under isometry.

Theorem (Gauss's Theorema Egregium)**.** *If (M, g) is a 2-dimensional submanifold of $\mathbb{R}^3$, where g is the induced metric, then for any $p \in M$ and any basis (X, Y) for T_pM, the Gauss curvature K is given by*

$$K = \frac{R(X,Y,Y,X)}{g(X,X)g(Y,Y) - g(X,Y)^2}$$

Since R, and g are invariant under isometry, so is K. We can now give a geometric interpretation of the sectional curvature. Suppose that (M, g) is a Riemannian manifold and $p \in M$. Let Π be a 2-dimensional plane in T_pM and let $V \subset T_pM$ be a neighbourhood of 0 such that $\exp_p$ is a diffeomorphism, then let $S_\Pi \equiv \exp_p(\Pi \cap V)$ be a 2-dimensional submanifold of M containing p. S_Π is called the **plane section** determined by Π. So S_Π can the thought of as the set of swept out by geodesics that start at p, with initial velocity in Π. We denote $K(\Pi)$ to be the Gauss curvature of S_Π, we then have the following relation.

Theorem. *If X, Y are a basis for $\Pi \subset T_pM$, then*

$$K(\Pi) = \frac{R(X,Y,Y,X}{g(X,X)g(Y,Y) - g(X,Y)^2},$$

Thus the sectional curvature encodes the Gauss curvature of all plane sections of M.

4 Decomposition of the curvature tensor

The Riemann curvature tensor encodes a lot of information, and although it has many symmetries, the fact that it is a $(0, 4)$-tensor makes it difficult to work with in general. We often want to summarize relevant information from the curvature to simplify expressions or to get the meaningful information without the clutter. Two of the most important tensors that encode information from the Riemann curvature tensor are the Ricci tensor, and the Weyl tensor. We will explain what they are, what they measure, and provide a geometric interpretation for each. Note we will be assuming the connection is Levi-Civita for the remaining of our discussion with curvature.

4.1 Ricci curvature

The Ricci curvature is an incredibly important quantity since it plays such an important role in the Einstein equation. It is essential to get a better geometric interpretation of what exactly it measures.

Definition. The **Ricci curvature tensor** Ric is a symmetric $(0, 2)$-tensor whose indices are defined by:

$$(\text{Ric})_{jk} \equiv R^i_{ijk},$$

where R^l_{ijk} are the indices of the Riemann curvature tensor. The **scalar curvature** R is define to be

$$R \equiv \mathrm{tr}(Rc^{\#}) = R_{jk}g^{jk}$$

We will first give a geometric interpretation of what Ric is. Let $X \in T_pM$ be a unit vector. We can choose an orthonormal basis for $\{X_1, \ldots, X_n\}$ for T_pM such that $X = X_1$, then:

$$\mathrm{Ric}(X, X) = R_{11} = R^i_{i11} = \sum_{i=1}^{n} R(X_i, X_1, X_1, X_i) = \sum_{i=2}^{n} K(X_i, X_1).$$

One can show that this is well defined in the sense that it is independent of the choice of orthonormal basis. So we have $\mathrm{Ric}(X, X)$ is the sum of sectional curvatures of the planes spanned by X. Since Ric is a symmetric bilinear form is, it is completely determined by $R(X, X)$.

Similarly if $\{X_1, \ldots, X_n\}$ is an orthonormal basis for T_pM, then

$$R = \sum_{i \neq j} K(X_i, X_j).$$

So we have $\mathrm{Ric}(X, X)$ is the sum of sectional curvatures of the planes spanned by orthogonal basis pairs.

Another interpretation of Ric is the following. Let $X \in T_pM$ for some $p \in M$. Let U be any neighbourhood of p. For each $q \in C$ we define $q_t \equiv \gamma_q(t)$ where γ_q is the geodesic starting at q with initial velocity X. We let $U_t \equiv \{q_t | q \in U\}$. It can be shown that [Oll13]

$$\mathrm{Vol}(U_t) = \mathrm{Vol}(U)\left(1 - \frac{t^2}{2}\mathrm{Ric}(X, X) + O(t^3)\right),$$

so Ric controls the change in volumes under a geodesic flow. The limitation is that it does not tell us how the shape of the object is changing. For example, we could have a ball undergo geodesic flow and have it stretch into an ellipsoid, but always have the same volume at all times, and the Ricci curvature tensor will not give any insight into how the shape is changing. To combat the issue we examine the Weyl tensor.

4.2 Weyl curvature

The Weyl tensor, like the Ricci tensor is defined in terms of the Riemann curvature tensor. However both tell the very different information about the curvature. We already showed that the Ricci curvature encodes information about volume, Weyl curvature tell us information about the shape.

Definition. Assume that $\dim(M) \geq 3$, then the **Weyl curvature** is defined as

$$C_{ijkl} = R_{ijkl} - \frac{2}{n-1}(g_{i[k}R_{l]j} - g_{j[k}R_{l]i}) + \frac{2}{(n-1)(n-2)}Rg_{i[k}g_{l]j}$$

Although this may seem complicated, the Weyl curvature has the same symmetries as the Riemann curvature tensor as described in the beginning of section 3, with the added condition that it is trace free. The Ricci curvature and Weyl curvature together encode the same information as the curvature tensor.

Theorem. Weyl tensor is that is conformally invariant, meaning, if $\tilde{g} = fg$, where f is a positive smooth function, then the $\tilde{C} = C$ where $\tilde{C}$ is the Weyl tensor under the conformal change in metric.

Since R vanished if and only if locally g is the euclidean metric, we have immediately by the above theorem that the Weyl tensor vanishes if and only if g is locally conformal to the euclidean metric. If the $Weyl$ curvature vanishes then we say that (M, g) is **conformally flat**.

Gravitational tidal forces, gravitational waves usually stretch is some directions and contract in other. Since the Ricci curvature cares about changes in volume, it doesn't capture their information unlike the Weyl tensor. For example as the moon travels around the earth, it does not change, volume, but different parts of the moon are experiencing different magnitude of the gravitational forces, the Weyl curvature tells us how object experiences the forces as it evolves. [Bae13]

5 Holonomy

Before we begin our discussion of holonomy we fix our notation by defining the parallel transport of a vector field along a smooth curve. The reader is expected to be familiar with these and their interpretation.

Definition. Let (M, ∇) be a manifold with a connection on TM, $p, q \in M$, $X_p \in T_pM$, and $\gamma : [0, 1] \to M$ be a smooth curve such that $\gamma(0) = p, \gamma(1) = q$. A tensor field $\tilde{X}$ along γ is called the **parallel transport** of X_p, if

1. $\tilde{X}_p = X_p$
2. $(D_t\tilde{X})(t) = 0$ for all $t \in [0, 1]$.

We will denote the **parallel transport** of along γ by $\Pi_\gamma : T_pM \to T_qM$

$$\Pi_\gamma(X_p) = \tilde{X}_q.$$

One can show that the parallel transport always exists and is unique, and that Π_γ is invertible. We know that one of the ways we measure shape of a manifold is by examining the non-triviality of the parallel transport. It is natural to ask how badly does it fail, and if so, can we use it to get more structure out of the manifold? The holonomy takes the set of all possible parallel transports and we can study the group they generate under composition. To study this group we do not need any structure other than a connection. Similar to connection, holonomy is a measure of how badly parallel transport fails. We now define the holonomy group of a connection.

Definition. Let (M, ∇) be a manifold with a connection on TM, $p \in M$, then the **holonomy group** of ∇ based at p, is

$$\mathrm{Hol}_p(\nabla) \equiv \{\Pi_\gamma \in \mathrm{GL}(T_pM) | \gamma : [0, 1] \to M, \text{ is a smooth curve such that} \gamma(0) = \gamma(1) = p\}$$

The **restricted holonomy group** based at p is the subgroup $\mathrm{Hol}^0_p(\nabla)$ of $\mathrm{Hol}_p(\nabla)$ of contractable loops based at p.

We note that $\mathrm{Hol}_p(\nabla)$ is indeed a group under composition. We will now remove the need to fix a base point, by showing that $\mathrm{Hol}_p(\nabla) \cong \mathrm{Hol}_q(\nabla)$ if p, q are in the same connected components of M. This is due to the fact that if τ is a curve that joints p, q, then $\Phi : \mathrm{Hol}_p(\nabla) \to \mathrm{Hol}_q(\nabla)$

$$\Phi(\Pi_\gamma) = \Pi_\tau \Pi_\gamma \Pi_\tau^{-1}.$$

It is easy to verify Φ is an isomorphism between the two groups, since we follow parallel transport from q to p via τ then along γ from p to p, and then τ in reverse. This results in a parallel transport from q to q. We will now stop mentioning the basepoint because all the groups are isomorphic up to conjugation. I will now list some facts about $\mathrm{Hol}(\nabla)$.

Proposition. [KN96] Let ∇ be a connection on TM, and let $\mathrm{Hol}(\nabla)$, $\mathrm{Hol}^0(\nabla)$ be the holonomy group and restricted holonomy group of ∇ respectively. Then

1. $\mathrm{Hol}^0(\nabla)$ is a connected Lie subgroup of $GL(n, \mathbb{R})$.
2. $\mathrm{Hol}^0(\nabla)$ is the identity component of $\mathrm{Hol}(\nabla)$, i.e. it is the connected component of $\mathrm{Hol}(\nabla)$ that contains the identity.
3. If M is simply connected, then $\mathrm{Hol}(\nabla) = \mathrm{Hol}^0(\nabla)$.
4. M is flat with respect to ∇ iff the curvature tensor R vanishes iff $\mathrm{Hol}^0(\nabla)$ is trivial.

The last point is important since it is tells us that holonomy is indeed a measure of how curved a space is. This is natural since the curvature tensor measures the failure of the parallel transport along an infinitesimal "parallelogram". However, a parallelogram is contractable if we make it small enough, and thus the parallel transport along it vanishes iff the curvature is 0 iff the parallel transport is trivial iff $\mathrm{Hol}^0(\nabla)$ is trivial. Although it measures the non-triviality of the parallel transport, it tells us very different information than the curvature. The curvature tensor gives us more of an analytic way to manipulate the non-trivialities of the parallel transport, whereas the holonomy gives us an algebraic one. Both are useful in their own right, but offer a different view on the same problem.

5.1 Riemannian holonomy

We now restrict our attention to the Levi-Civita connection. In this case we call $\mathrm{Hol}(\nabla)$ the **Riemannian holonomy**. In the case that ∇ is the Levi-Civita connection we can say a lot more about the structure of $\mathrm{Hol}(\nabla)$.

Theorem (Borel & Lichnerowicz)**.** *[KN96] The Riemannian holonomy group is a compact Lie subgroup of* $\mathrm{SO}(n, \mathbb{R})$*.*

The following theorem shows that in the case of the Riemannian holonomy, the curvature tensor and the holonomy group are closely related.

Theorem (Ambrose-Singer theorem)**.** *[KN96] If M is simply connected, then the Lie-algebra $\mathfrak{h}$ of the Riemannian holonomy* $\mathrm{Hol}_p(\nabla)$ *is generated as a vector space by the curvature operator $R(X, Y)$ for all $X, Y \in T_pM$.*

Note that the above result is much more general, but it requires a lot more machinery to introduce than is relevant for this exposition. Ambrose-Singer says states that the holonomy group is generated by the curvature operator $R(X, Y)$, meaning they encode a lot of the same information.

5.2 DeRham decomposition

We will need to use basic facts from representation theory for this section.

Definition. If G is a group, and V is a finite dimensional vector space, we say that that V is a **representation** of G is there is exists a group action $\rho : G \to GL(V)$. If $W \subset V$ be a subspace, we say that W is a **sub-representation** of V if W is invariant under the action of G, i.e. $\rho(q)(W) \subset W, \forall g \in G$. Finally we say that a representation is **irreducible** if the only sub-representations are are $\{0\}$ and V.

One result from representation theory we will use is if W is a sub-representation of V, then so is $W^\perp$, the orthogonal complement of W in V, and $V = W \bigoplus W^\perp$. So we can repeat this process always decompose V as the direct sum of irreducible sub-representations. The holonomy group $\mathrm{Hol}_p(\nabla)$ act naturally by left multiplication on to T_pM, so T_pM is a representation of $\mathrm{Hol}_p(\nabla)$. So we have $T_pM = \bigoplus_{i=0}^k T_p^iM$ where T_p^iM is an irreducible sub-representation of T_pM. We also say that $T_p^0M = \{X \in T_pM | \Pi_\gamma(X) = X, \forall \Pi_\gamma \in \mathrm{Hol}_p(\nabla)\}$ is the trivial sub-representation of T_pM possible 0-dimensional. The following theorem tells us how the representations of the holonomy group can give us insight on the local properties of a Riemannian manifold.

Theorem (de Rham decomposition)**.** [KN96] Let (M, g) be a simply connected Riemannian manifold with the Levi-Civita connection ∇. Let $T_pM = \bigoplus_{i=0}^k T_p^iM$ be the representation of $\mathrm{Hol}_p(\nabla)$, where T_p^iM are irreducible, and T_p^0M, let T^iM be the involutive distribution on M obtained by parallel transport of T_p^iM. By Frobineus theorem [Lee03], there is a corresponding integral manifold M^i, such that M^i satisfies $T_pM^i = T_p^iM$. We have the following:

1. M^0 is a euclidean space of dimension $\dim(T_p^0M)$.

2. M is locally isometric to $V^0 \times V^1 \times \cdots \times V^k$, where V^i is on open subset of M^i.

3. $\mathrm{Hol}_p(\nabla)$ is the direct product of the holonomy groups of each of the M_i.

4. If M is complete (i.e. there are no singularities then), then so are M^i and M is isometric to $M^0 \times M^1 \times \cdots \times M^k$.

Thus the holonomy group gives a way to study the local structure of a Riemannian manifold.

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