Numerical explorations of the dynamics of FRW cosmologies

Saifuddin Syed, #20387835
University of Waterloo
Submitted to Dr. Achim Kempf for AMATH 875

December 19, 2013

Abstract

We begin our exploration of the FRW cosmologies by deriving the Friedman equations. We then explore both analytically and numerically evolution of the scale factor in different epochs, including the cases where the universe has non-zero curvature. After that we discuss the condition required to have a universe where the scale parameter shrinks to a finite value, non-zero value, an continues to expand forever. We end our exploration with a discussion of the evolutions of the scale parameter under the presence of multiple types of matter.

Before we begin our exploration, we should note that all the diagrams and computations were done using Maple 16, and lecture notes were extensively used throughout this entire project, and will only be cited here [Kem13].

1 Derivation of Friedman equations

The evolution of the universe in FRW spacetime is governed by the Friedman equations. We begin by briefly outlining how these equations are derived. In the FRW universe, it is assumed that the universe is modelled as a Lorentzian 4-manifold (M, g) which can be decomposed into the following form:

$$M = I \times \Sigma,$$
$$g = -dt^2 + a(t)^2 \overline{g},$$

where $I \subset \mathbb{R}$ is an interval, $t \in I$ is called **cosmic time**, and $a(t)$ is called the **scale factor**, which is a smooth function of M, that is spatially constant. $(\Sigma, \overline{g})$ is a fully isotropic, and homogeneous Riemannian 3-manifold. Note that these conditions imply that Σ has constant sectional curvature K.

Spaces of constant sectional curvature have a very rigid structure:

- If $K > 0$, then Σ is isometric to a 3-sphere and Σ is closed.
- If $K = 0$, then Σ is isometric to $\mathbb{R}^3$ and Σ is flat.
- If $K < 0$, then Σ is isometric to 3-dimensional hyperbolic space.

To derive the Friedman equations we first define a tetrad locally around a point $p = (t_0, p_0) \in M$ by the following equations:

$$\theta^0 \equiv dt,$$
$$\theta^i \equiv a(t)\overline{\theta}^i,$$

where $\overline{\theta}$ is an orthonormal triad locally around $p_0 \in (\Sigma, \overline{g})$. We note that θ^i form an orthonormal tetrad for (M, g). Our goal now is the derive the curvature 2-form M.

To do so, we calculate $d\theta^i$ in two ways.

1. By applying the definition of d and the 1^{st} structure equation for Σ we get:

$$d\theta^i = -\frac{\dot{a}}{a}\theta^i \wedge \theta^0 - \overline{\omega}^i_j \wedge \theta^i. \tag{1}$$

2. By directly applying the 1^{st} structure equation to $d\theta^i$ on M to get:

$$d\theta^i = -\omega^i_0 \wedge \theta^0 - \omega^i_j \wedge \theta^j. \tag{2}$$

Equating 1,2 results in the following identity:

$$\omega^i_0 = \frac{\dot{a}}{a}\theta^i \quad \text{and} \quad \omega^i_j = \overline{\omega}^i_j. \tag{3}$$

Now we compute the curvature 2-form for M. By applying the second structure equation when $1 \leq i, j \leq 3$, we obtain:

$$\begin{aligned} \Omega^i_j &= d\omega^j_i + \omega^i_\mu \wedge \omega^\mu_j \\ &= \overline{\Omega}^i_j + \omega^i_0 \wedge \omega^0_j. \end{aligned} \tag{4}$$

Since Σ is a space of constant sectional curvature K, the curvature 2-form is given by

$$\overline{\Omega}^i_j = K\overline{\theta}^i \wedge \overline{\theta}^j = \frac{K}{a^2}\theta^i \wedge \theta^j. \tag{5}$$

Substituting 3,5 into 4 and simplifying results in

$$\Omega^i_j = \frac{K + \dot{a}^2}{a^2}\theta^i \wedge \theta^j. \tag{6}$$

Using an analogous argument we can compute Ω^0_i.

$$\Omega^0_i = \frac{\ddot{a}}{a}\theta^0 \wedge \theta^i. \tag{7}$$

Combining 6,7 with the fact that $\Omega_{\mu\nu} = \frac{1}{2}R_{\mu\nu\sigma\xi}\theta^\sigma \wedge \theta^\xi$, we can obtain the curvature tensor $R_{\mu\nu\sigma\xi}$. From the curvature tensor, we can find the Ricci tensor $R_{\mu\nu}$,= and the scalar curvature R by summing of the appropriate indices of the curvature tensor. Substituting these into the Einstein tensor $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$ gives us

$$G_{00} = 3\left(\frac{\dot{a^2} + K}{a^2}\right), \quad G_{ii} = -2\frac{\ddot{a}}{a} - \frac{\dot{a}}{a^2} - \frac{K}{a^2}. \tag{8}$$

The off-diagonal entries are 0 since $g_{\mu\nu}$ and $R_{\mu\nu}$ are both diagonal in the θ^i frame. So the Einstein equation tells us that $G_{\mu\nu} = 8\pi G T_{\mu\nu}$, implying the energy-momentum tensor $T_{\mu\nu}$ is also diagonal. We know that the entries of $T_{\mu\nu}$ are as follows:

$$T_{00} = \rho, \quad T_{ii} = p, \tag{9}$$

where ρ is the **matter energy density** and p is the **matter pressure**. Note that we have incorporated the cosmological constant into the definition of ρ and p. Substituting 8,9 into the Einstein equation we get the Friedman equations:

$$3\left(\frac{\dot{a}^2}{a^2} + \frac{K}{a^2}\right) = 8\pi G\rho \tag{10}$$

$$-2\frac{\ddot{a}}{a} - \frac{\dot{a}}{a^2} - \frac{K}{a^2} = 8\pi G p. \tag{11}$$

We can clean this up a bit by eliminating the $\dot{a}$ term from 11 to get

$$\ddot{a} = -\frac{4\pi G a}{3}(\rho + 3p). \tag{12}$$

2 Evolution during epochs

Recall that an **epoch** is a period of time where the equation of state parameter $w(\rho)$ is constant. Ie. we have a linear dependency between pressure and density,

$$p(\rho) = w\rho. \tag{13}$$

As shown in class we have that the Friedman equations implies

$$\dot{\rho} = -3\frac{\dot{a}}{a}(\rho + p), \tag{14}$$

or equivalently,

$$\frac{d}{da}(\rho a^3) = -3pa^2. \tag{15}$$

Substituting in (13) into (15) and simplifying we get,

$$\frac{d\rho}{da} = -3\frac{\rho}{a}(w + 1). \tag{16}$$

To solve 16, we apply separation of variables to get ρ as a function of the scale factor,

$$\rho(a) = \rho_0 a^{-3(w+1)}. \tag{17}$$

To keep physical we will assume that ρ_0 is non negative. Using 17, 10 becomes,

$$\frac{\dot{a}^2}{a^2} + \frac{K}{a^2} = \frac{8\pi G}{3}\rho_0 a^{-3(w+1)} \tag{18}$$

To solve this equation for the different epochs we need to look at the case where the curvatures are different.

2.1 $K = 0$

The simplest case is when we assume the universe is spatially flat, thus $K = 0$. In this case 18 simplifies significantly:

$$\dot{a} = \pm\sqrt{\frac{8\pi G\rho_0}{3}}a^{-\frac{3(w+1)}{2}+1} = \pm C a^{-\frac{3(w+1)}{2}+1}, \tag{19}$$

where $C = \sqrt{\frac{8\pi G\rho_0}{3}}$. Again we can solve for a by apply separation of variables to 19:

$$a(t) = \begin{cases} \left(\pm\frac{3(w+1)C}{2}t + A\right)^{\frac{2}{3(w+1)}}, & \text{if } w \neq -1 \\ Ae^{\pm Ct}, & \text{if } w = -1 \end{cases} \tag{20}$$

here A is a constant that depends on initial conditions, and whether we chose $\pm$ depends $\dot{a}(t_0)$. Let's examine 20 in more detail.

2.2 $w \geq -1$

When $w > -1$, the solutions all contain a singularity when $a = 0$. This signifies that these epochs must have come into existence as a result of a "big bang", and then continued to expand forever. Since we are assuming that $a > 0$ and time is going forward, we can conclude that the universe in these epochs continues to increase indefinitely. The rate at which $a(t)$ increases is $O\left(t^{\frac{2}{3(w+1)}}\right)$.

In particular, in the case of matter-dominated universe ($w = 0$) we have that a is increasing at a rate of $O\left(t^{\frac{2}{3}}\right)$, and in a radiation-dominated universe ($w = \frac{1}{3}$) we have that a increases at a rate of $O\left(t^{\frac{1}{2}}\right)$.

When $w = -1$, i.e. in the case where the cosmological constant dominates, we have that a grows (decays) exponentially if $\dot{a}$ is positive (negative). One point of note is that in this epoch we do not have a singularity as a is never 0. So this epoch does not suggest the existence of a big bang.

Figure 1 contrast how the scale factor evolves during each of the epochs.

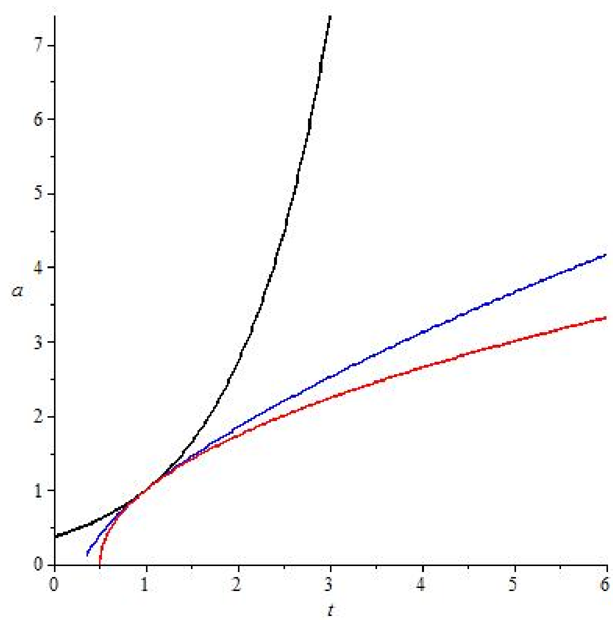

Figure 1: The diagram shows how a evolves during cosmological constant (black), matter-dominated (blue), radiation-dominated (red) epochs under the same initial conditions.

2.3 Curved spacetime

We now consider the case where the universe has non-negative scalar curvature, so $K \neq 0$. To find how the scaling factor evolves during each epoch we need to solve 18, with $K \neq 0$. As in the flat universe case, we need to apply separation of variables to solve for a; because of the curvature term, it is very difficult to solve analytically. We will perform some qualitative analysis and numerical methods. First note that 18 and 12 tell us,

$$|\dot{a}| = \sqrt{\frac{8\pi G}{3}\rho_0 a^{-3w-1} - K} \tag{21}$$

$$\ddot{a} = -\frac{4\pi G a}{3}\rho(1+3w). \tag{22}$$

For any given epoch, as K increases, $|\dot{a}|$ decreases, so the greater the curvature the slower a will grow or decay for any given epoch. So higher curved space acts as a retardant on the expansion of the universe, since it resists rapid growth or decay. Regardless of the curvature, we have that 22 implies $\ddot{a} > 0$ when $w < -\frac{1}{3}$ and $\ddot{a} < 0$ when $w > -\frac{1}{3}$. First let us assume that $w < -1/3$. In this case case 21 implies that $a \to \infty$, which implies $\dot{a}$ does as well. When a is large, the curvature term is negligible and we have $\dot{a} \sim \sqrt{\frac{8\pi G}{3}\rho_0 a^{-3w-1}}$, meaning that when a is large the rate at which it grows is the same regardless of curvature. Since we already know the rate when $K = 0$, we have that regardless of curvature a grows at a rate of $O\left(t^{\frac{2}{3(w+1)}}\right)$ when $-1 < w < -1/3$, and exponentially when $w = -1$.

If $K > 0$, then $|\dot{a}|$ needs to always be positive. If a is increasing then 21 implies $|\dot{a}|$ will also increase as long as a^{-3w-1} increases with a. This is only possible if $-3w - 1 > 0$ or $w < -\frac{1}{3}$. If $w > -\frac{1}{3}$ then a^{-3w-1} decreases to 0 as $a \to \infty$. Eventually $|\dot{a}|$ will reach 0 for some finite a. When $w > -1/3$ we have that $\ddot{a}$ is negative, meaning that when $|\dot{a}| = 0$, a is at a local maximum, and thus a will collapse into a "big crunch" in some finite time. When $w = -\frac{1}{3}$ if $K = \frac{8\pi G}{3}\rho_0$ then a will be constant, and if $K > \frac{8\pi G}{3}\rho_0$, then a will increase (or decrease) linearly with time.

If $K < 0$ then 21 implies $|\dot{a}| > 0$ for all a so either a will always increase with time or decrease (depending on initial conditions), and will not switch signs. When $w \geq -\frac{1}{3}$ we have that as a approaches infinity $\dot{a}$ approaches $\sqrt{K}$. So in the limit a increases a rate of $O(t)$.

Figure 2 shows how $a(t)$ evolves with time, for the different epochs with the same initial conditions.

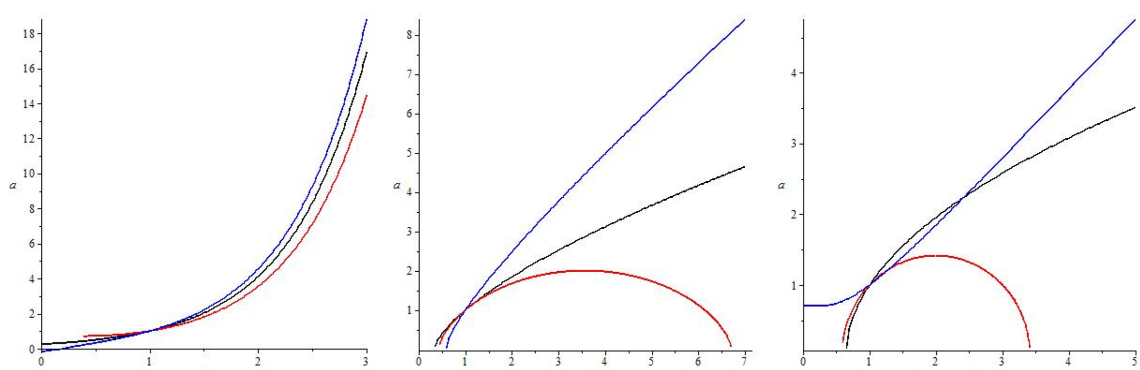

Figure 2: The diagrams shows how a evolves during cosmological constant (left), matter-dominated (middle), radiation-dominated epochs(right) under the same initial conditions when $K = -1$(blue), $K = 0$(black), and $K = -1$(red).

2.4 w less than -1, oh no!

We recall that $w \in [-1, 1]$ for all known forms of matter. However recent data indicated that currently for our universe $w = -1.19$. Let us examine in further detail what an epoch with an equation of parameter below -1 means for the evolution of $a(t)$. We first note that when $w < -1$ then 17 implies that

$$\rho(a) = \rho_0 a^{\varepsilon}, \tag{23}$$

for some $\epsilon > 0$. 23 implies that as the universe expands, the density also increases. This is very counter-intuitive, as one expects the universe to dilute as a increases. This implies that as time increases we will approach an infinite density, and expansion. In the case of a flat universe we have that 20 implies that

$$a(t) = \frac{1}{(\pm At + B)^{\epsilon}}, \tag{24}$$

for some constants A, B, and $\epsilon > 0$. This implies that for some finite time there is a singularity. Solutions for the different curvatures are plotted below.

In the case of different curvatures, our remark in the previous section regarding $|\dot{a}|$ with different curvatures did not use the fact that $w \geq -1$, so regardless of curvature we have that a increases at the same rate. This implies that regardless of the curvature we have a singularity at some finite time. In general as K increases, then $|\dot{a}|$ decreases, which implies the singularity occurs later for higher curvatures, compared to lower ones, as seen in Figure 3.

If this is indeed the universe we are in and there are no phase transitions causing w to remain below -1, then our universe will eventually be "ripped" apart by the unknown matter and we have an expiry date.

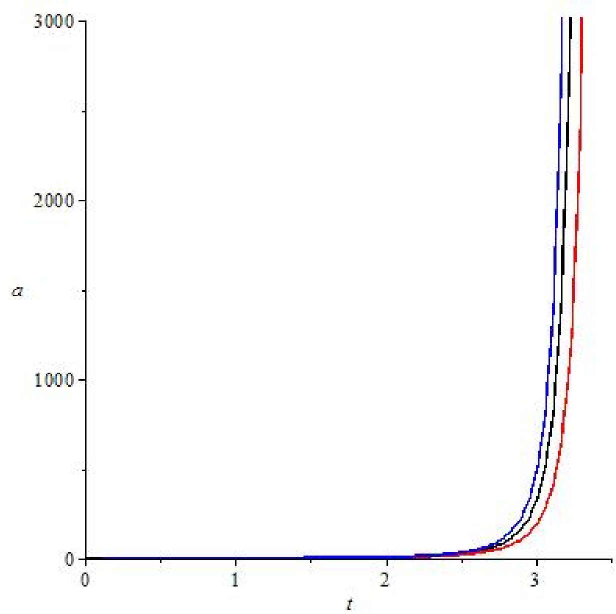

Figure 3: The diagrams shows how a when $w = -1.19$ under the same initial conditions when $K = -1$(blue), $K = 0$(black), and $K = -1$(red).

3 Boing!

We have already seen examples of universes where the scale parameter expands and then eventually collapses in some finite time. However in every case we have seen so far, we note that a collapses back down to 0, resulting in a singularity, or a "big crunch". A reasonable question to ask, is if it is possible for the scale parameter to decrease to a finite non-zero value, and then expand once again, a so called "bounce"? The answer turns out to be yes, and we will outline the necessary conditions for $w(\rho)$.

We begin by noting that for a bounce to occur there needs to exist time t_0 such that $\dot{a}(t_0) = 0$ and $\ddot{a}(t_0) > 0$ locally at t_0, ie a local minimum. Under these conditions 10,11 imply the following.

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3}\rho - \frac{K}{a^2} = 0, \tag{25}$$

$$\ddot{a} = \frac{-4\pi G a \rho}{3}(1+3w) > 0. \tag{26}$$

First lets look at the case where the universe is spatially flat and $K = 0$. Then we have that 25 implies that $\rho(t_0) = 0$, but then 26 implies that $p(t_0) < 0$. Since $w = \frac{p}{\rho} \to -\infty$ as $t \to t_0$ which clearly cannot happen since $w \in [-1, 1]$. So we cannot have that in a flat universe a bounce cannot happen.

If there is curvature present to have 25 satisfied, then for a bounce to occur at t_0 we need $\rho(t_0)$ and K to have the same sign, and $\rho \propto \frac{K}{a^2}$. Note that in the case of negative curvature, 25 can only be satisfied if the density is negative, or equivalently the cosmological constant is negative and dominates at time t_0. Now if $K > 0$ then $\rho_0 \equiv \rho(t_0) > 0$, which implies that $1+3w < 0$ or $w < -\frac{1}{3}$. Similarly, when $K < 0$ we need $w > -\frac{1}{3}$.

Recall during an epoch we have that 17 tells us that $\rho \propto a^{-3(w+1)}$. Setting $-3(w+1) = -2$ we find that $w = -\frac{1}{3}$. So the problem has been reduced to finding $w(\rho)$ that satisfies the following criterion:

- $w(\rho_0) = -\frac{1}{3}$.
- w is approximately constant at ρ_0, so that w approximates an epoch.
- w need to be strictly less (greater) than $-\frac{1}{3}$ when the curvature is positive (negative) locally at ρ_0 for the concavity to be positive

One such valid $w(\rho)$ is the following:

$$w(\rho) = \begin{cases} -\frac{1}{3} - \mathrm{sgn}(K)\frac{2}{3}\exp\left(-\frac{1}{(\rho-\rho_0)^2}\right), & \text{if } \rho \neq \rho_0 \\ -\frac{1}{3}, & \text{if } \rho = \rho_0 \end{cases}, \tag{27}$$

where $\mathrm{sgn}(K)$ is 1 is $K > 0$ and is -1 when $K < 0$. This particular 27 is between $(-1, -\frac{1}{3}]$ in the positive curvature case and between $[-\frac{1}{3}, \frac{1}{3})$ in the negative curvature case. It is smooth, but not analytic at ρ_0 since every derivative is 0 at ρ_0. Thus this function is approximately "constant" at ρ_0 but is locally either less than (greater than) $-\frac{1}{3}$ in the case of positive (negative) curvature. See figure 4 for the bounce. Mathematically speaking, such an equation of state parameter is possible for some choice of scalar field and potential; whether it is physically possible or likely is beyond the scope of this paper.

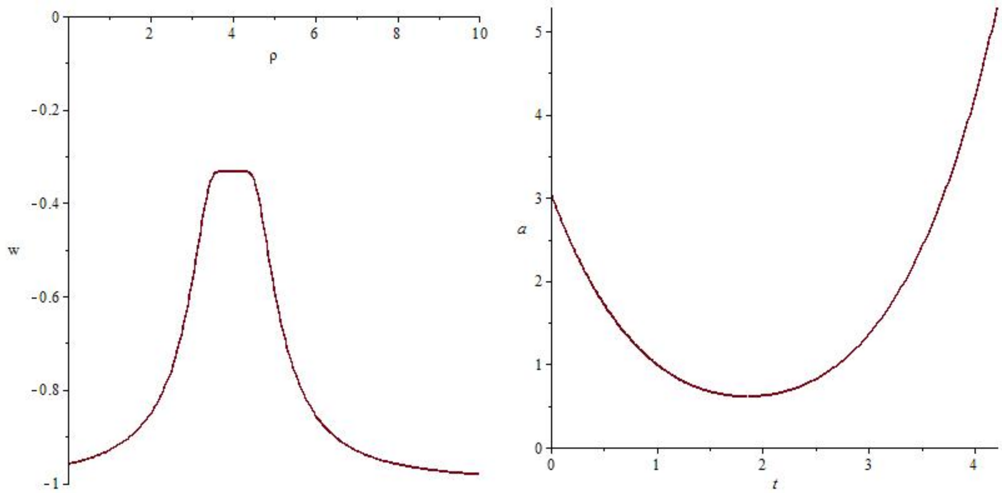

Figure 4: The left is the $w(\rho)$ defined at the end of section 3. The right is $a(t)$ in a universe with an equation of state parameter as described in the left diagram.

4 Evolution of the scale factor

[Car03] Now we shift our attention to a universe with multiple types of matter. We begin by defining the **Hubble parameter** H by

$$H \equiv \frac{\dot{a}}{a}.$$

We also define the **critical density** and the **density parameter** respectively as

$$\rho_{crit} \equiv \frac{3H^2}{8\pi G}, \quad \Omega \equiv \frac{\rho}{\rho_{crit}}. \tag{28}$$

So substituting 4 into 10, the Friedman equation becomes

$$H^2 = \frac{8\pi G}{3}\rho - \frac{K}{a^2} \tag{29}$$

or equivalently by dividing through by H^2, and substituting 28 into 29,

$$\Omega - 1 = \frac{K}{H^2 a^2}. \tag{30}$$

This then gives us an equivalent measure of when the curvature is positive, zero, or negative, summarized below:

$$\begin{aligned} K > 0 &\Leftrightarrow \Omega > 1, \\ K = 0 &\Leftrightarrow \Omega = 1, \\ K < 0 &\Leftrightarrow \Omega < 1. \end{aligned} \tag{31}$$

In order to analyze how a evolves under the presence of multiple forms of matter, we make the following simplifying assumptions:

1. The universe has three forms of densities: vacuum, matter, and radiation, denoted by ρ_Λ, ρ_m, and ρ_r. We pressure for each density is $p_i = w_i \rho_i$, (where $w_\Lambda = -1, w_m = 0$, and $w_r = \frac{1}{3}$). In other words we have

$$\begin{aligned} \rho &= \rho_\Lambda + \rho_m + \rho_r, \\ p &= w_\Lambda \rho_\Lambda + w_m \rho_m + w_r \rho_r. \end{aligned} \tag{32}$$

2. These forms of matter do not interact with each other. In other words there are no phase transitions.

3. Finally we will assume that each ρ_i will evolve as if they were evolving in their respective epoch. Therefore their evolution is governed by 17, or equivalently,

$$\rho_i = \rho_{i0} a^{-3(w_i+1)}, \tag{33}$$

where ρ_{i0} are constants.

The first assumption is very fair since those were the main forms of matter that dominate the universe as it evolves. So to find ρ, p can be considered the total density and pressure. Assumption 2 is a poor one to make for modelling early universe as there was a large amounts of phase transition from radiation to matter. However our main goal in this section is to explore how the scale factor will evolve in the future, and we know that the universe now is primarily dominated by matter and the cosmological constant, with negligible radiation. Assumption 3 is a fair one to make because since we are assuming that the different forms of matter are not interacting, they each evolve how they would in their own epoch. Now we define

$$\Omega_i \equiv \frac{\rho_i}{\rho_{crit}}. \tag{34}$$

With this definition we have $\Omega = \Omega_\Lambda + \Omega_m + \Omega_r$. To simplify notation we also define

$$\Omega_K \equiv -\frac{K}{H^2 a^2}, \quad \rho_K \equiv -\frac{3K}{8\pi G a^2}. \tag{35}$$

With this notation we can now write the Friedman equation as

$$1 = \Omega + \Omega_K. \tag{36}$$

If we have the initial condition that $a(t_0) = 1$. Also we can define units so that $H(t_0)^2 = \frac{8\pi G}{3}$. Thus we have

$$\rho(t_0) = \sum_i \rho_{i0} = \frac{8\pi G}{3H(t_0)^2} \sum_i \rho_{i0} = \sum_i \Omega_i(t_0) = 1$$

Where the sum is over $i = \Lambda, K, m, r$. So keeping $\rho_{K0} = -\frac{3K}{8\pi G}$ constant, and letting $\rho_{m0}, \rho_{\Lambda 0}, \rho_{r0}$ vary, completely determines the behaviour of the system. So we just need to specify ρ_{i0} for $i = \Lambda, m, r$.

We will now assume the universe is completely dominated by the cosmological constant, and matter, with radiation being negligible compared to the two (similar to our universe right now). Thus we have $\rho_{r0} \approx 0$, and

$$\rho_{K_0} = 1 - \rho_{\Lambda 0} - \rho_{m0}. \tag{37}$$

So in this universe the entire evolution of a is purely determined by the pair $(\rho_{m0}, \rho_{\Lambda 0}) \in [0, \infty) \times \mathbb{R}$, since we are assuming ρ_m is non-negative.

We have $\Omega(t_0) = \rho_{\Lambda 0} + \rho_{m0}$. By 31 we have that if $\rho_{\Lambda 0} + \rho_{m0} > 1$ then $K < 0$, if $\rho_{\Lambda 0} + \rho_{m0} = 1$ then $K = 0$, and if $\rho_{\Lambda 0} + \rho_{m0} < 1$ then $K > 0$. Also want to find the conditions where the universe will expand forever or eventually contract. To answer this equation we need to find when $\dot{a} = 0$ for some positive a. If $\rho_{\Lambda 0}$ is negative we will always have $H = 0$ in some finite time as a gets large enough. When $\rho_{\Lambda 0} > 0$ we have to do a bit more work. Substituting in $H = 0$ in 29 we get the following cubic equation:

$$0 = \rho(a) = \rho_{m0}a^{-3} + \rho_{K0}a^{-2} + \rho_{\Lambda 0}$$

or equivalently,

$$0 = \rho_{m0} + (1 - \rho_{\Lambda 0} - \rho_{m0}a + \rho_{\Lambda 0}a^3$$

[Car03] Solving this cubic and finding restrictions to when the root is positive, we get $a(t) \to \infty$ if

$$\rho_{\Lambda 0} \geq \begin{cases} 0, & 0 \leq \rho_{m0} \leq 1 \\ 4\rho_{m0}\cos^3\left(\frac{1}{3}\arccos\left(\frac{1-\rho_{m0}}{\rho_{m0}}\right) + \frac{4\pi}{3}\right), & \rho_{m0} > 1 \end{cases},$$

and collapses in a finite time otherwise. figure 5 shows some samples of the scale parameter $(\rho_{m0}, \rho_{\Lambda 0}$, and summarizes the results.

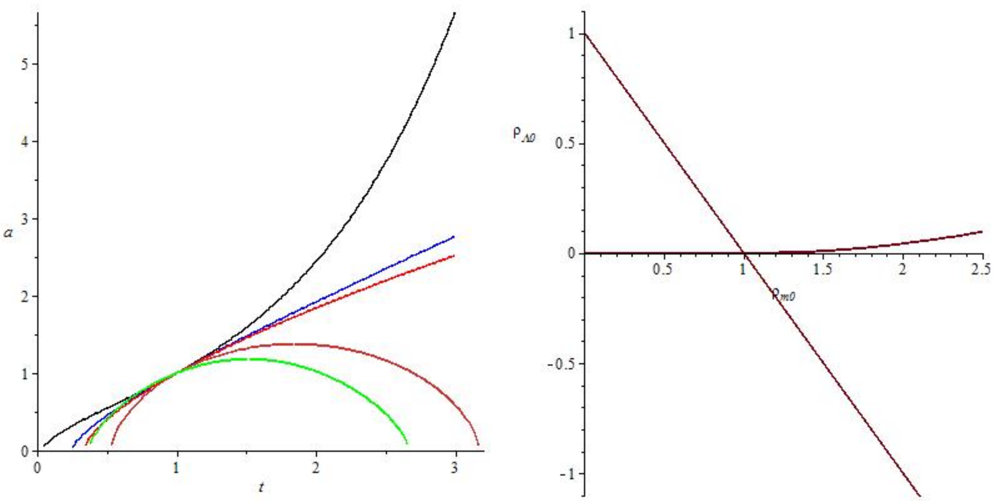

Figure 5: The **first** figure shows plots $a(t)$ for various $(\rho_{m0}.\rho_{\Lambda 0})$. The universe we are currently in (.3,.7) is in black. (.5,0),(1,0),(4,0.1),(2,-0.5) are in blue, red, orange, and green respectively. The **second** figure shows the curature and long term behaviour of a for various inition conditions of $(\rho_{m0}.\rho_{\Lambda 0})$. a will expand forever, if $(\rho_{m0}.\rho_{\Lambda 0})$ iff if $(\rho_{m0}.\rho_{\Lambda 0})$ are on or above the flat curve. Points above the line $\rho_{m0} + \rho_{\Lambda 0} = 1$ imply the universe has positive curvature. Points below the line $\rho_{m0} + \rho_{\Lambda 0} = 1$ imply the universe has negative curvature. Points on the line $\rho_{m0} + \rho_{\Lambda 0} = 1$ imply the universe is flat.

References

[Car03] Sean Carroll. *Spacetime and Geometry: An Introduction to General Relativity*. Benjamin Cummings, 2003.

[Kem13] Achim Kempf. General relativity for cosmology fall 2013, 2013.