Spatial diffusions with singular drifts: The construction of Super Brownian motion with immigration at unoccupied sites

by

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Abstract

In this essay we explored two original problems related to diffusions with singular drifts. In Chapter 1, we construct diffusions X on $[0, \infty)$ using speed and scale analysis, that satisfy the stochastic differential equation (SDE),

$$X_t = X_0 + \int_0^t X_s^p dB_s + bt,$$

for $0 < p \leq \frac{1}{2}$, $b \geq 0$, when $X > 0$, but exhibit sticky boundary behaviour at 0. I.e., X spends a positive amount time at zero, but never a full interval. We then find the SDE characterization of the sticky diffusions and show that in the case where $p = \frac{1}{2}$, the SDE fails to uniquely encode the boundary behaviour of the process.

In Chapter 2, we then proceed to the realm of spatial diffusions and construct a non-trivial continuous measure-valued process X that solves the stochastic partial differential equation (SPDE),

$$\frac{\partial X}{\partial t} = \frac{\Delta X}{2} + \sqrt{X}\dot{W} + \dot{A},$$

where W is a space-time white noise and A is a continuous measure-valued immigration such that A_t is only active in the regions where X_t does not occupy mass.

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Preface

Over the course of my masters, I have worked on a variety of problems in stochastic analysis and diffusions. In this essay we will work through two of them; each self-contained in their own chapter.

In Chapter 1, we study a class of 1 dimensional diffusions on X on $[0, \infty)$ which satisfy the stochastic differential equation (SDE),

$$X_t = X_0 + \int_0^t X_s^p dB_s + bt,$$

when $X > 0$, for $p \leq \frac{1}{2}$, $b \geq 0$, and exhibit sticky boundary behaviour at 0. By sticky boundary behaviour, we mean that the process spends a positive time at 0 but never a full interval. We will construct the diffusions using scale and speed analysis and determine when their SDE representation uniquely encodes the sticky boundary behaviour.

In Chapter 2, we turn our attention to spatial diffusions. Motivated by population genetics, we construct a population on $\mathbb{R}$, undergoing random motion and critical reproduction, under the influence of immigration at unoccupied sites. In particular, we construct a measure-valued diffusion X satisfying the stochastic partial differential equation (SPDE),

$$\frac{\partial X}{\partial t} = \frac{\Delta X}{2} + \sqrt{X}\dot{W} + \dot{A}.$$

Where W is a space-time white noise and A is a continuous measure-valued process which is only supported at the location where X occupies no population. This will be formalized later.

Contents

Chapter 1

SDE characterization of diffusions with sticky boundaries

In Theorem 8 of (Burdzy et al., 2010), Burdzy, Mueller, and Perkins showed that when $p < \frac{1}{2}$, $b > 0$, non-negative solutions to the stochastic differential equation (SDE),

$$X_t = X_0 + \int_0^t X_u^p dB_u + bt, \tag{1.1}$$

exhibit Feller's sticky boundary behaviour at 0. I.e,

$$\int_0^t 1(X_s = 0)ds > 0.$$

Our goal in this chapter is to strengthen their result and classify all diffusion on X on $[0, \infty)$ that satisfy (1.1) when when $X > 0$ and exhibit Feller's sticky boundary behaviour at 0. We will do this by following the method of Ito and McKean outlined in (Knight, 1981, § 6,7).

We will first examine the case where $p < \frac{1}{2}$. We note that we separating the drift bt into time spent by the process away from 0, and the spend at 0 to rewrite (1.1) as,

$$X_t = X_0 + \int_0^t X_u^p dB_u + b\int_0^t 1(X_s > 0)ds + b\int_0^t 1(X_s = 0)ds. \tag{1.2}$$

Section 1.1, will show that the stickiness of X is controlled by the coefficient of the $1(X = 0)$ term in the drift in (1.2). We will use speed and scale analysis to argue that the SDE fully characterize the sticky boundary of our process.

In the case of $p = \frac{1}{2}$, after scaling by a constant (1.1) is equivalent to the equation,

$$X_t = X_0 + \int_0^t 2\sqrt{X_u} dB_u + \alpha t, \tag{1.3}$$

which is the SDE of a $\text{Bess}^2(\alpha)$ process. We will construct a family of sticky $\text{Bess}^2(\alpha)$ processes and find their corresponding SDE. We will then argue that in this case the SDE representation is not robust enough to uniquely encode the boundary behaviour.

1.1 Characterization of sticky behaviour for $0 < p < \frac{1}{2}$

Let X_t be a non-negative solution to the SDE,

$$X_t = X_0 + \int_0^t X_s^p dB_s + b\int_0^t 1(X_s > 0)ds + \rho \int_0^t 1(X_s = 0)ds, \tag{1.4}$$

where $0 < p < \frac{1}{2}$, and $b \geq 0$, $\rho > 0$. Note that in the special case where $b = \rho$, we get (1.1). Our aim for this section is to prove the following:

1) X exhibits sticky boundary behaviour at 0 governed by ρ.

2) The SDE encodes the boundary behaviour as described by the scale function and speed measure.

We will show 1) by analysing the scale function, and speed measure for X, and 2) by show that any diffusion on $[0, \infty)$ with the same scale function and speed measure as X, must satisfy (1.4).

We now proceed with the first step of the program.

1.1.1 Existence of sticky boundary behaviour

Proposition 1.1.1. *If X is a solution to* (1.4)*, then it is the diffusion on* $[0, \infty)$ *with the scale function,*

$$s(x) = \int_0^x \exp\left\{\frac{-2b|y|^{1-2p}}{1-2p}\right\} dy, \tag{1.5}$$

and, speed measure m on $[0, s(\infty))$ defined by,

$$m(dx) = \frac{dx}{s'(s^{-1}(x))^2 s^{-1}(x)^{2p}} + \frac{1}{\rho}\delta_0(dx). \tag{1.6}$$

Moreover if $T_0 = \inf\{t | X_t = 0\}$, then for all $\varepsilon > 0$,

$$\int_{T_0}^{T_0+\varepsilon} 1(X_s = 0)ds > 0. \tag{1.7}$$

Remark 1.1.2. Note that the scale function can be found by solving the ode in terms of the drift and volatility as described in (Rogers and Williams, 2000, §V.28). Also since $b > 0$ and $0 < p < \frac{1}{2}$, we have s is increasing with $s(\infty) < \infty$ and has an inverse $s^{-1} : [0, s(\infty)) \to [0, \infty)$.

The proof below is a modification of the proof of Theorem 8 in (Burdzy et al., 2010).

Proof. We begin by analysing the scale function,

$$s'(x) = \exp\left\{\frac{-2b|x|^{1-2p}}{1-2p}\right\}.$$

So when $x > 0$ we have the s'' is given by,

$$s''(x) = \exp\left\{\frac{-2b|x|^{1-2p}}{1-2p}\right\}(-2bx^{-2p}) = -2bs'(x)x^{-2p}.$$

Define $L_t^X(x)$ as the local time process of X, and let $Y_t = s(X_t)$. So we have by Ito-Tanaka's formula,

$$\begin{aligned} Y_t = &\ s(X_t) \\ = &\ Y_0 + \int_0^t s'(X_s)X_s^p dB_s + b\int_0^t s'(X_s)1(X_s > 0)ds \\ &+ \rho\int_0^t s'(X_s)1(X_s = 0)ds + \frac{1}{2}\int L_t^X(x)ds'(x). \end{aligned}$$

Now let's simplify the local time term. Since s' is continuous, we have ds' has no atoms.

$$\begin{aligned}
\frac{1}{2}\int L_t^X(x)ds'(x) &= \frac{1}{2}\int 1(x>0)L_t^X(x)ds'(x) \\
&= \frac{1}{2}\int 1(x>0)L_t^X(x)s''(x)dx \\
&= \frac{1}{2}\int 1(x>0)L_t^X(x)(-2bs'(x)x^{-2p})dx \\
&= -b\int_0^t 1(X_s>0)s'(X_s)X_s^{-2p}X_s^{2p}ds \\
&= -b\int_0^t 1(X_s>0)s'(X_s)ds.
\end{aligned}$$

Thus we have

$$Y_t = Y_0 + \int_0^t s'(X_s)X_s^p dB_s + \rho\int_0^t s'(X_s)1(X_s=0)ds.$$

Let us define

$$U \equiv \langle Y\rangle_\infty = \int_0^\infty s'(X_s)^2 X_s^{2p}ds.$$

Define the random time change $\alpha : [0,U) \to [0,\infty)$ as the solution to the equation

$$\int_0^{\alpha(t)} s'(X_s)^2 X_s^{2p}ds = t. \tag{1.8}$$

It is clear that since the integrand is non-negative we have that α is increasing and positive. Since X is non zero on any interval, α is continuous. Define $R_t = Y_{\alpha(t)}$, we will now show that R is a reflecting Brownian motion starting at Y_0 for $t < U$, and can be extended to in the natural way for $t \geq U$. We have for $t < U$,

$$\begin{aligned}
R_t &= Y_0 + \int_0^{\alpha(t)} s'(X_s)X_s^p dB_s + \rho\int_0^{\alpha(t)} 1(X_s=0)ds \\
&= Y_0 + \beta_t + A_t.
\end{aligned}$$

Where $\beta_t = Y_0 + \int_0^{\alpha(t)} X_s^p dB_s$, and $A_t = \rho\int_0^{\alpha(t)} 1(X_s=0)ds$. Note that for $t < U$ we have by (1.8),

$$\langle\beta\rangle_t = \int_0^{\alpha(t)} s'(X_s)^2 X_s^{2p}ds = t.$$

So β is a stopped Brownian motion starting at Y_0, and A is a continuous non-decreasing and supported by $\{t : X_{\alpha(t)} = 0\}$ or equivalently $\{t < U : R_t = 0\}$. Thus by the Skorokod problem (Rogers and Williams, 2000, §V.6), R is a reflecting Brownian motion and for $t < U$,

$$\rho \int_0^{\alpha(t)} 1(X_s = 0) ds = A_t = L_t^R(0). \tag{1.9}$$

Note that (1.9) implies (1.7). Thus it remains to find the speed measure of X. Since

$$\int_0^t s'(X_s)^2 X_s^{2p} ds = \alpha^{-1}(t),$$

by the fundamental theorem of calculus,

$$(\alpha^{-1})'(t) = s'(X_t)^2 X_t^{2p}. \tag{1.10}$$

We now note that by (1.9) and (1.10),

$$\begin{aligned} t &= \int_0^t 1(X_s > 0) ds + \int_0^t 1(X_s = 0) ds \\ &= \int_0^t \frac{1(X_s > 0)}{s'(X_s)^2 X_s^{2p}} d(\alpha^{-1}(s)) + \frac{1}{\rho} L^R_{\alpha^{-1}(t)}(0). \end{aligned}$$

If we let $s = \alpha(u)$, we get

$$\begin{aligned} t &= \int_0^{\alpha^{-1}(t)} \frac{1(X_{\alpha(u)} > 0)}{s'(X_{\alpha(u)})^2 X_{\alpha(u)}^{2p}} du + \frac{1}{\rho} L^R_{\alpha^{-1}(t)}(0) \\ &= \int_0^{\alpha^{-1}(t)} \frac{1(R(u) > 0)}{s'(s^{-1}(R_u))^2 s^{-1}(R_u)^{2p}} du + \frac{1}{\rho} L^R_{\alpha^{-1}(t)}(0) \\ &= \int_0^{s(\infty)} L^R_{\alpha^{-1}(t)}(x) \left[\frac{1(x > 0) dx}{s'(s^{-1}(x))^2 s^{-1}(x)^{2p}} + \frac{1}{\rho} \delta_0(dx) \right]. \end{aligned}$$

Therefore the speed measure of X is

$$m(dx) = \frac{1(x > 0) dx}{s'(s^{-1}(x))^2 s^{-1}(x)^{2p}} + \frac{1}{\rho} \delta_0(dx),$$

on $[0, s(\infty))$ as required. So we can conclude that

$$X_t = s^{-1}(R_{\alpha^{-1}(t)}),$$

is the diffusion on $[0,\infty)$ with scale function s and speed measure m, where

$$t = \int_0^{s(\infty)} L^R_{\alpha^{-1}(t)}(x)m(dx).$$

□

Therefore we have shown that solutions to the SDE (1.4) exhibit sticky boundary behaviour at 0. The goal now is to show that the converse to Proposition 1.1.1 holds as well.

1.1.2 Differentiability of random time change

In order to show the converse of Proposition 1.1.1, we will be forced to work with various random time changes by $\tau(t)$, which is determined as the solution to the equation,

$$t = \int_0^{\tau(t)} f(B_s)ds + \frac{1}{\rho}L_t,$$

where B is a standard Brownian motion with local time L, and sufficiently f positive and "nice".

The goal of this section is to show that τ is a.s. differentiable and to compute it's derivative.

Lemma 1.1.3. *Let L be the local time of Brownian motion, then,*

$$\liminf_{h\to 0} \frac{L_h}{h} = \infty,$$

Proof. Let's show that $\mathbb{P}(L_{2^{-n}} < \phi(2^{-n})\ i.o.) = 0$, for $\phi(h) = h^{1/2}(\log(1/h))^{-1-\varepsilon}$, where $\varepsilon > 0$. So indeed, using the fact that L_t has the same law as $\sup_{s\le t} B_s$, and the reflection principle we have,

$$
\begin{aligned}
\mathbb{P}(L_{2^{-n}} < \phi(2^{-n})) &= \mathbb{P}(|B_{2^{-n}}| < \phi(2^{-n})) \\
&= \mathbb{P}(\sqrt{2^{-n}}|B_1| < \sqrt{2^{-n}}(\log(2^n)^{-1-\varepsilon}) \\
&= \mathbb{P}\left(|B_1| < \frac{(\log 2)^{-1-\varepsilon}}{n^{1+\varepsilon}}\right) \\
&= 2\int_0^{\frac{(\log 2)^{-1-\varepsilon}}{n^{1+\varepsilon}}} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz \\
&\leq 2\frac{(\log 2)^{-1-\varepsilon}}{n^{1+\varepsilon}}.
\end{aligned}
$$

Which is summable. Thus by Borel-Cantelli, $\mathbb{P}(L_{2^{-n}} < \phi(2^{-n}) i.o.) = 0$ and

$$
\liminf_{h\to 0} \frac{L_h}{\phi(h)} = \liminf_{n\to\infty} \frac{L_{2^{-n}}}{\phi(2^{-n})} \geq 1.
$$

Where the first equality is using the fact that L and ϕ are continuous, along with an interpolation argument. Now we note that

$$
\liminf_{h\to 0} \frac{L_h}{h} = \liminf_{h\to 0} \frac{L_h}{\phi(h)} \frac{\phi(h)}{h} \geq \lim_{h\to 0} \frac{\phi(h)}{h} = \infty.
$$

□

Proposition 1.1.4. *Let f be a non-negative integrable function such that for all $\eta > 0$,*

$$
f_\eta \equiv \inf_{\eta < |x| < \eta^{-1}} f(x) > 0.
$$

Define $\tau(t)$ to be the continuous inverse of,

$$
A(t) = \int_0^t f(B_s) ds + \frac{1}{\rho} L_t,
$$

Where L_t is the local time of Brownian motion. Then the following holds:

(a) $\tau(t)$ is a.s. absolutely continuous with respect to the Lebesgue measure.

(b) If f is continuous when $x > 0$, then the Radon-Nikodym derivative equals

$$
\tau'(t) = \frac{1}{f'(B_{\tau_t})} 1(B(\tau_t) > 0).
$$

Proof.

(a) Since f is non-negative, and τ is the inverse of A, we have τ is increasing and satisfies,

$$t = \int_0^{\tau(t)} f(B_s)ds + \frac{1}{\rho} L_{\tau_t}. \tag{1.11}$$

We will show that $\tau(t)$ is absolutely continuous with respect to Lebesgue measure on $[0, T]$ for all $T > 0$. Let $\varepsilon > 0$ and suppose $[s_i, t_i] \subset [0, T]$ satisfying

$$\sum_i |t_i - s_i| < \delta.$$

Now note that,

$$\begin{aligned}
&\sum_i |\tau(t_i) - \tau(s_i)| \\
=&\sum_i \int_{\tau(s_i)}^{\tau(t_i)} du \\
=&\sum_i \int_{\tau(s_i)}^{\tau(t_i)} 1(|B_u| \leq \eta)du + \sum_i \int_{\tau(s_i)}^{\tau(t_i)} 1(\eta < |B_u| < \eta^{-1})du \\
&+ \sum_i \int_{\tau(s_i)}^{\tau(t_i)} 1(|B_u| \geq \eta^{-1})du \\
\leq& \int_0^{\tau(T)} 1(|B_u| \leq \eta)du + \sum_i \int_{\tau(s_i)}^{\tau(t_i)} 1(\eta < |B_u| < \eta^{-1})du \\
&+ \int_0^{\tau(T)} 1(|B_u| \geq \eta^{-1})du \\
\equiv& I_1 + I_2 + I_3
\end{aligned}$$

Since B_t is continuous on $[0, T]$, we have that there exists $\eta(\omega)$ small enough such that $B_t(\omega) \leq \eta(\omega)^{-1}$. Therefore $I_3 = 0$. Because Brownian motion occupies zero time at zero, we can pick η small enough such that $I_1 < \frac{\varepsilon}{2}$. So it remains to bound I_2. Let $\delta < \frac{f_\eta \varepsilon}{2}$, then we have

$$
\begin{aligned}
I_2 =& \sum_i \int_{\tau(s_i)}^{\tau(t_i)} 1(\eta < |B_u| < \eta^{-1}) du \\
\leq& \sum_i \int_{\tau(s_i)}^{\tau(t_i)} \frac{f(B_u)}{f_\eta} 1(\eta < |B_u| < \eta^{-1}) du \\
\leq& \frac{1}{f_\eta} \sum_i \int_{\tau(s_i)}^{\tau(t_i)} f(B_u) du \\
=& \frac{1}{f_\eta} \sum_i (t_i - L_{\tau(t_i)}) - (s_i - L_{\tau(s_i)}) \quad \text{, by (1.11)} \\
\leq& \frac{1}{f_\eta} \sum_i t_i - s_i \\
\leq& \frac{1}{f_\eta} \delta \\
<& \frac{\varepsilon}{2}
\end{aligned}
$$

Thus we have shown τ is absolutely continuous with respect to Lebesgue measure. By Radon-Nikodym theorem, there exists a $\tau'(t)$ such that

$$
\tau(t) = \int_0^t \tau'(u) du.
$$

(b) By rearranging (1.11), we see that for $h > 0$,

$$
1 = \frac{\int_{\tau_t}^{\tau_{t+h}} f(B_s) ds}{h} + \frac{1}{\rho} \frac{L_{\tau_{t+h}} - L_{\tau_t}}{h}.
$$

Suppose that $B_{\tau_t} = 0$, so the second term is non-zero. Since $f \geq 0$ we have the inequality

$$
1 \geq \frac{1}{\rho} \frac{L_{\tau_{t+h}} - L_{\tau_t}}{h}.
$$

By rearranging terms we have,

$$
\begin{aligned}
\rho \geq& \frac{L_{\tau_{t+h}} - L_{\tau_t}}{h} \\
=& \frac{L_{\tau_{t+h}} - L_{\tau_t}}{\tau_{t+h} - \tau_t} \frac{\tau_{t+h} - \tau_t}{h} \\
=& \frac{L_{\tau_h}}{\tau_h} \circ \theta_{\tau_t} \frac{\tau_{t+h} - \tau_t}{h}. \qquad (1.12)
\end{aligned}
$$

Where θ_t is the left shift by t. The last equality was applying the shift to the Markov process $\frac{L_{\tau_t}}{\tau_t}$ by the stopping time τ_t. The strong Markov property tells us the first term of (1.12) has the same law as $\frac{L_{\tau_h}}{\tau_h}$. So by lemma 1.1.3, the liminf is infinity, which implies the limit is also infinity as h goes to zero. So for all N there is a $\delta > 0$ such that $h < \delta$ implies,

$$\frac{L_{\tau_h}}{\tau_h} \circ \theta_{\tau_t} > N.$$

Therefore we have,

$$\frac{\rho}{N} \geq \frac{\tau_{t+h} - \tau_t}{h}.$$

Letting N go to infinity gives us that the right derivative of $\tau(t)$ both exists and equals 0. Since τ is absolutely continuous with respect to Lebesgue measure, we have the left limit also exists and equals the right. So when $B_{\tau_t} = 0$, we have $\tau'(t) = 0$.

Now Suppose that $B_{\tau_t} > 0$, then $L_{\tau_{t+h}} - L_{\tau_t} = 0$ for h small enough. So we have

$$1 = \frac{\int_{\tau_t}^{\tau_{t+h}} f(B_s)ds}{h} = \frac{\int_{\tau_t}^{\tau_{t+h}} f(B_s)ds}{\tau_{t+h} - \tau_t} \frac{\tau_{t+h} - \tau_t}{h}.$$

Therefore by rearranging the terms we get,

$$\lim_{h \to 0} \frac{\tau_{t+h} - \tau_t}{h} = \lim_{h \to 0} \frac{\tau_{t+h} - \tau_t}{\int_{\tau_t}^{\tau_{t+h}} f(B_s)ds} = \frac{1}{f(B_{\tau_t})}.$$

The last equality was by the fundamental theorem of calculus. Thus we have shown that

$$\tau'(t) = \frac{1}{B_{\tau_t}} 1(B_{\tau_t} > 0).$$

□

1.1.3 Equivalence of SDE to speed/scale

Equipped with Proposition 1.1.4, we are ready to prove the converse to Proposition 1.1.1.

Proposition 1.1.5. *Suppose X is a diffusion on $[0, \infty)$ with s, and speed measure m defined on $[0, s(\infty))$ given by* (1.5), *and* (1.6) *respectively. Then X is a solution to* (1.4).

Proof. Let R be a reflected Brownian motion with initial law $s(X_0)$ with local time L^R. We have by the definition of scale function and speed measure that

$$X_t = s^{-1}(R_{\tau_t}),$$

where τ_t is the continuous inverse of

$$\begin{aligned} A(t) &= \int_0^{s(\infty)} L_t^R(x) m(dx) \\ &= \int_0^{s(\infty)} \frac{L_t^R(x) 1(x > 0)}{s'(s^{-1}(x))^2 s^{-1}(x)^{2p}} dx + \frac{1}{\rho} L_t^R(0) \\ &= \int_0^t \frac{1(R_u > 0)}{s'(s^{-1}(R_u))^2 s^{-1}(R_u)^{2p}} du + \frac{1}{\rho} L_t^R(0). \end{aligned} \tag{1.13}$$

Since $s(x) = x + o(x)$, and $s'(x) = 1 + o(x)$, we have $s^{-1}(x) = x + o(x)$ and

$$\frac{1(x > 0)}{s'(s^{-1}(x))^2 s^{-1}(x)^{2p}} \approx \frac{1(x > 0)}{x^{2p}},$$

Which satisfies the criteria of Proposition 1.1.4, thus

$$\tau'(t) = s'(X_t)^2 X_t^{2p} 1(X_t > 0). \tag{1.14}$$

Now to get an SDE representation of X we apply Ito-Tanaka's formula.

$$\begin{aligned} X(t) &= s^{-1}(R_{\tau_t}) \\ &= s^{-1}(R_0) + \int_0^{\tau_t} (s^{-1})'(R_u) dR_u + \frac{1}{2} \int L_{\tau_t}^R(x) d(s^{-1})'(x) \\ &= s^{-1}(s(X_0)) + \int_0^{\tau_t} \frac{1}{s'(s^{-1}(R_u))} d(\beta_u + L_u^R(0)) + \frac{1}{2} \int_0^{\tau_t} (s^{-1})''(R_u) du \\ &= X_0 + \int_0^{\tau_t} \frac{1}{s'(s^{-1}(R_u))} d\beta_u + \int_0^{\tau_t} \frac{1}{s'(s^{-1}(R_u))} dL_u^R(0) + \frac{1}{2} \int_0^{\tau_t} (s^{-1})''(R_u) du \\ &\equiv X_0 + M_t + I_t^1 + I_t^2. \end{aligned} \tag{1.15}$$

Where β_t is a Brownian motion. We will have to take care of each of the three terms individually. Let's begin with analysing the quadratic variation of M:

$$\begin{aligned}[M]_t &= \int_0^{\tau_t} \frac{1}{s'(s^{-1}(R_u))^2 du} \\ &= \int_0^t \frac{1}{s'(s^{-1}(R_{\tau_v}))^2} \tau'(v) dv \\ &= \int_0^t \frac{1}{s'(X_v)^2} s'(X_v)^2 X_v^{2p} 1(X_v > 0) dv \\ &= \int_0^t X_v^{2p} 1(X_v > 0) dv \\ &= \int_0^t X_v^{2p} dv.\end{aligned}$$

The third line was because of (1.14). Define

$$\tilde{B} = \int_0^t \frac{1(X_u > 0)}{X_u^p} dM_u + \int_0^t 1(X_u = 0) d\tilde{\beta}_u,$$

where $\tilde{\beta}$ is an independent Brownian motion and note

$$\langle \tilde{B} \rangle_t = \int_0^t \frac{1(X_u > 0)}{X_u^{2p}} X_u^{2p} du + \int_0^t 1(X_u = 0) du = t.$$

So by Lévy's characterization of Brownian motion, there exists a Brownian motion $\tilde{B}$, such that

$$M_t = \int_0^t X_v^p d\tilde{B}_v. \tag{1.16}$$

We can finally deal with the finite variation terms.

$$\begin{aligned}I_t^1 &= \int_0^{\tau_t} \frac{1}{s'(s^{-1}(R_u))} dL_u^R(0) \\ &= \int_0^{\tau_t} \frac{1}{s'(s(0))} dL_u^R(0) \\ &= \int_0^{\tau_t} dL_u^R(0) \\ &= L_{\tau_t}^R(0).\end{aligned}$$

We now note that by rearranging (1.13) we get,

$$
\begin{aligned}
I_t^1 &= \rho \left(t - \int_0^{\tau_t} \frac{1(R_u > 0)}{s'(s^{-1}(R_u))^2 s^{-1}(R_u)^{2p}} du \right) \\
&= \rho \left(t - \int_0^{t} \frac{1(R_{\tau_v} > 0)}{s'(s^{-1}(R_{\tau_v}))^2 s^{-1}(R_{\tau_v})^{2p}} \tau'(v) dv \right) \\
&= \rho \left(t - \int_0^{t} \frac{1(X_v > 0)}{s'(X_v)^2 X_v^{2p}} s'(X_v)^2 X_v^{2p} dv \right) \\
&= \rho \left(t - \int_0^{t} 1(X_v > 0) dv \right) \\
&= \rho \int_0^t 1(X_v = 0) dv. \qquad (1.17)
\end{aligned}
$$

Before we simplify I^2, let's first compute $(s^{-1})''(x)$, and we use the fact that when $x > 0$, we have $s''(x) = -2bs'(x)x^{-2p}$. So for $x > 0$,

$$
\begin{aligned}
(s^{-1})''(x) &= \left(\frac{1}{s'(s^{-1}(x))} \right)' \\
&= -\frac{[s'(s^{-1}(x))]'}{s'(s^{-1}(x))^2} \\
&= -\frac{s''(s^{-1}(x))}{s'(s^{-1}(x))^3} \\
&= \frac{2bs'(s^{-1}(x))(s^{-1}(x))^{-2p}}{s'(s^{-1}(x))^3} \\
&= \frac{2b}{s'(s^{-1}(x))^2 s^{-1}(x)^{2p}}.
\end{aligned}
$$

As $(s^{-1})'$ is continuous at x, we have

$$
\begin{aligned}
I_t^2 &= \frac{1}{2} \int_0^{\tau_t} \frac{2b}{s'(s^{-1}(R_u))^2 s^{-1}(R_u)^{2p}} du \\
&= \frac{1}{2} \int_0^{t} \frac{2b}{s'(s^{-1}(R_{\tau_v}))^2 s^{-1}(R_{\tau_v})^{2p}} \tau'(v) dv \\
&= b \int_0^t \frac{1}{s'(X_v)^2 X_v^{2p}} 1(X_v > 0) s'(X_v)^2 X_v^{2p} dv \\
&= b \int_0^t 1(X_v > 0) dv. \qquad (1.18)
\end{aligned}
$$

By substituting (1.16), (1.17),(1.18) into (1.15), we have shown that

$$
X_t = X_0 + \int_0^t X_v^p d\tilde{B}_v + \rho \int_0^t 1(X_v = 0) dv + b \int_0^t 1(X_v > 0) dv,
$$

as required. $\square$

This implies that for the case where $p < \frac{1}{2}$, (1.4) encodes the same information as the scale function and speed measure.

1.2 Non-uniqueness of SDE for sticky Bess$^2(\alpha)$ process

In the case where $p = \frac{1}{2}$, we have (1.1) is equivalent to the SDE for the Bess$^2(\alpha)$ process given by,

$$X_t = X_0 + \int_0^t 2\sqrt{X_u}dB_u + \alpha t,$$

for $\alpha > 0$. The analysis of the Bess$^2(\alpha)$ process in (Rogers and Williams, 2000, §V.48) shows that 0 is recurrent if and only if $0 < \alpha < 2$, but it does not exhibit any sticky boundary behaviour, i.e., for all $t > 0$,

$$\int_0^t 1(X_u = 0)du = 0.$$

Following the method of Ito and KcKean as described in (Knight, 1981, § 6,7), we can construct a Bess$^2(\alpha)$ process with sticky boundary behaviour at 0, by applying the same construction as in the case of $p < \frac{1}{2}$.

It was shown in (Rogers and Williams, 2000, §V.48) that the Bess$^2(\alpha)$ process has scale function $s(x) = x^{\frac{2-\alpha}{2}}$, and speed measure $\tilde{m}$ defined on $[0, \infty)$ given by

$$\tilde{m}(dx) = \frac{1}{(2-\alpha)^2}x^{\frac{2\alpha-2}{2-\alpha}}1(x > 0)dx.$$

So if X is the Bess$^2(\alpha)$ process with sticky boundary behaviour at 0, then the X also has the scale function s and speed measure m given by

$$m(dx) = \tilde{m}(dx) + \frac{1}{\rho}\delta_0(dx),$$

where $\rho > 0$ controls how "sticky" 0 is. We will now find an SDE representation of X, the sticky Bess$^2(\alpha)$ process.

Proposition 1.2.1. *The process X on $[0,\infty)$ with the scale function and speed measure on $[0,\infty)$, given by*

$$s(x) = x^{\frac{2-\alpha}{2}},$$

and,

$$m(dx) = \frac{1}{(2-\alpha)^2} x^{\frac{2\alpha-2}{2-\alpha}} 1(x>0)dx + \frac{1}{\rho}\delta_0(dx),$$

respectively, satisfies the SDE:

$$X_t = X_0 + \int_0^t 2\sqrt{X_u}dB_u + \alpha \int_0^t 1(X_u > 0)du. \tag{1.19}$$

Proof. Let B be a Brownian motion with initial law $s(X_0) = X_0^{\frac{2-\alpha}{2}}$, and $L_t(x)$ be the local time of the reflected Brownian motion $|B|$. Let's define

$$\begin{aligned} A(t) &= \int_0^\infty L_t(x)m(dx) \\ &= \int_0^\infty L_t(x)\left(\frac{1}{(2-\alpha)^2} x^{\frac{2\alpha-2}{2-\alpha}} 1(x>0)dx + \frac{1}{\rho}\delta_0(dx)\right) \\ &= \int_0^t \frac{1}{(2-\alpha)^2}|B_u|^{\frac{2\alpha-2}{2-\alpha}} 1(|B_u|>0)du + \frac{1}{\rho}L_t(0). \end{aligned}$$

Let τ_t denote the right continuous inverse of A, ie, τ_t satisfies,

$$t = \int_0^{\tau_t} \frac{1}{(2-\alpha)^2}|B_u|^{\frac{2\alpha-2}{2-\alpha}} 1(|B_u|>0)du + \frac{1}{\rho}L_{\tau_t}(0).$$

Let $Y_t = |B_{\tau_t}|$ be the scaled sticky $\text{Bess}^2(\alpha)$ process. So we have

$$X = s^{-1}(Y_t) = |B_{\tau_t}|^{\frac{2}{2-\alpha}} \tag{1.20}$$

is the unscaled $\text{Bess}^2(\alpha)$. Since $|x|^p$ is convex and has an integrable second derivative when $p > 1$, and $\frac{2}{2-\alpha} > 1$, we can apply Tanaka's formula for $|x|^{\frac{2}{2-\alpha}}$ to (1.20).

$$\begin{aligned} X_t &= |B_{\tau_t}|^{\frac{2}{2-\alpha}} \\ &= (X_0^{\frac{2-\alpha}{2}})^{\frac{2}{2-\alpha}} + \int_0^{\tau_t} \frac{2}{2-\alpha}\text{sgn}(B_u)|B_u|^{\frac{\alpha}{2-\alpha}}dB_u + \frac{1}{2}\int_0^{\tau_t} \frac{2}{2-\alpha}\frac{\alpha}{2-\alpha}|B_u|^{\frac{2\alpha-2}{2-\alpha}}du \\ &\equiv X_0 + M_t + I_t. \end{aligned} \tag{1.21}$$

Now let us simplify each term individually, but first we will need $\tau'(t)$. By the Proposition 1.1.4,

$$\begin{aligned}\tau'(t) &= (2-\alpha)^2 |B_{\tau_t}|^{\frac{2-2\alpha}{2-\alpha}} 1(|B_{\tau_t}| > 0) \\ &= (2-\alpha)^2 Y_t^{\frac{2-2\alpha}{2-\alpha}} 1(Y_t > 0) \\ &= (2-\alpha)^2 (X_t^{\frac{2-\alpha}{2}})^{\frac{2-2\alpha}{2-\alpha}} 1(X_t > 0) \\ &= (2-\alpha)^2 X_t^{1-\alpha} 1(X_t > 0). \end{aligned} \tag{1.22}$$

To determine the quadratic variation of M, we will make the substitution $u = \tau(v)$ and using (1.22)

$$\begin{aligned}\langle M\rangle_t &= \int_0^{\tau_t} \frac{4}{(2-\alpha)^2} |B_u|^{\frac{2\alpha}{2-\alpha}} du \\ &= \int_0^t \frac{4}{(2-\alpha)^2} |B_{\tau_v}|^{\frac{2\alpha}{2-\alpha}} \tau'(v) dv \\ &= \int_0^t \frac{4}{(2-\alpha)^2} (X_v^{\frac{2-\alpha}{2}})^{\frac{2\alpha}{2-\alpha}} (2-\alpha)^2 X_v^{1-\alpha} 1(X_v > 0) dv \\ &= \int_0^t 4X_v 1(X_v > 0) dv. \end{aligned}$$

Define

$$\beta_t = \int_0^t \frac{1(X_v > 0)}{2\sqrt{X_v}} dM_v + \int_0^t 1(X_v = 0) dW_v,$$

where W is an independent Brownian motion. β has quadratic variation,

$$\langle \beta\rangle_t = \int_0^t \frac{1(X_v > 0)}{4X_v} 4X_v dv + \int_0^t 1(X_v = 0) dv = t.$$

Therefore β is a Brownian motion and

$$M_t = \int_0^t 1(X_v > 0) 2\sqrt{X_v} d\beta_v = \int_0^t 2\sqrt{X_v} d\beta_v. \tag{1.23}$$

Finally we simplify I_t, using the substitution $u = \tau_v$ and using (1.22).

$$\begin{aligned} I_t &= \frac{1}{2}\int_0^{\tau_t} \frac{2\alpha}{(2-\alpha)^2}|B_u|^{\frac{2\alpha-2}{2-\alpha}} du \\ &= \int_0^t \frac{\alpha}{(2-\alpha)^2}|B_{\tau_v}|^{\frac{2\alpha-2}{2-\alpha}} \tau'(v) dv \\ &= \int_0^t \frac{\alpha}{(2-\alpha)^2}(X_v^{\frac{2-\alpha}{2}})^{\frac{2\alpha-2}{2-\alpha}}(2-\alpha)^2 X_v^{1-\alpha} 1(X_v > 0) dv \\ &= \alpha\int_0^t 1(X_v > 0) dv. \end{aligned} \tag{1.24}$$

Finally by substituting (1.23), and (1.24) into (1.21) we conclude, that X satisfies the SDE,

$$X_t = X_0 + \int_0^t 2\sqrt{X_v} d\beta_v + \alpha\int_0^t 1(X_v > 0) dv.$$

□

For each $\rho > 0$, let X^ρ be the diffusion on $[0, \infty)$ with scale function $s(x) = x^{\frac{2-\alpha}{2}}$, and speed measure,

$$m^\rho(dx) = \frac{1}{(2-\alpha)^2} x^{\frac{2\alpha-2}{2-\alpha}} 1(x > 0) dx + \frac{1}{\rho}\delta_0(dx).$$

Proposition 1.2.1 tells us that each X^ρ satisfies (1.19). Therefore we have proven the following.

Corollary 1.2.2. *The solutions to the SDE* (1.19) *are not unique in law.*

This analysis for the $\text{Bess}^2(\alpha)$ process shows that SDE's are not in general best tool to analyse the boundary behaviour of diffusion.

1.3 Summary

In Chapter 1, we first began by analysing the the SDE,

$$X_t = X_0 + \int_0^t X_s^p dB_s + b\int_0^t 1(X_s > 0) ds + \rho\int_0^t 1(X_s = 0) ds. \tag{1.25}$$

for $0 < p < \frac{1}{2}$, and $b, \rho > 0$. This was motivated (Burdzy et al., 2010) where they studied the same SDE when $\rho = b$, where they discovered interesting sticky boundary behaviour.

In Section 1.1, we showed that non-negative solution X to (1.26) exhibits sticky boundary behaviour governed by ρ. We did this in Proposition 1.1.1 by analysing the scale function and speed measure of X. We went a step further and showed in Proposition 1.1.5 that any diffusion on $[0, \infty)$ with the same scale function and speed measure as X must also be a solution to (1.26).

Afterwards we moved our attention to the case where $p = \frac{1}{2}$, to the $\text{Bess}^2(\alpha)$ for $0 < \alpha < 2$. In Section 1.2, we followed the construction of Ito and McKean to construct a family of diffusions X^ρ on $[0, \infty)$ that behaved like $\text{Bess}^2(\alpha)$ with sticky boundary behaviour at 0 governed by ρ. In Proposition 1.2.1, we found that each X^ρ was a solution to the SDE,

$$X_t = X_0 + \int_0^t 2\sqrt{X_s} dB_s + \alpha \int_0^t 1(X_s > 0) ds. \tag{1.26}$$

This showed that the SDE's in general are not robust enough to encode the boundary behaviour of diffusions, and demonstrates the utility of speed and scale analysis.

Chapter 2

Super Brownian motion with immigration at unoccupied sites

Let us examine (1.4) in the case where $b = 0$. This results in the SDE,

$$X_t = \int_0^t X_s^p ds + \rho \int_0^t 1(X_s = 0)dx. \tag{2.1}$$

Note that X is a non-trivial diffusion on $[0, \infty)$ with a continuous immigration only active when the process is zero. Using (2.1) as a motivation, we will explore this idea for spatial processes.

A one-dimensional super-Brownian motion (SBM), X is used in population genetics to model a population on a line undergoing random motion and critical reproduction. Formally, X_t is a random finite measure on $\mathbb{R}$ changing continuously in t, and represents how the total population is dispersed at time t. (Perkins, 2002, §III.1,4) showed that at each time t, the population X_t has compact support, and X has satisfies the stochastic partial differential equation (SPDE),

$$\frac{\partial X}{\partial t} = \frac{\Delta X}{2} + \sqrt{X}\dot{W}.$$

where W is space-time white noise. (Perkins, 2002, §III.1) also showed that for all t, the population X_t is a.s. compactly supported. We are interested in seeing what happens to the population represented by SBM when we allow a continuous immigration only at the locations where the population does not occupy space. Our goal in this chapter will be to construct such a process and classify it.

We will define the spatial process X^ε as the sum of independent SBM populations X^i of initial mass ε generated at some random point x_i generated by a distribution $\psi(x)dx$ at time t_i via a Poisson process with temporal rate $\frac{1}{\varepsilon}dt$. We will only allow X^i to contribute to X^ε when the previously contributing populations do not occupy space at x_i. Note as ε gets smaller, each SBM cluster X^i will potentially add ε mass to X^ε but there will be proportionally $\frac{1}{\varepsilon}$ clusters contributing. This suggests that as $\varepsilon \to 0$, our process should converge to some non-trivial X.

The plan of attack will be as follows; after making our construction of X^ε more formal, we will then show that $\{X^\varepsilon\}$ are tight as $\varepsilon \to 0$. This will give us the existence of a continuous measure-valued process X. We will then show that X is non-trivial and satisfies a SPDE of the form,

$$\frac{\partial X}{\partial t} = \frac{\Delta X}{2} + \sqrt{X}\dot{W} + \dot{A}.$$

Where A is a continuous measure-valued immigration that is only active when the process X occupies no space. Without further ado, let us begin our construction.

2.1 Notation

Let Ξ be a $(\mathcal{F}_t)$-Poisson point process (PPP) on $\mathbb{R}_+ \times \mathbb{R}^d$ with rate $\Lambda(dt, dx) = \lambda dt \psi(x)dx$, where $\lambda > 0$, and ψ is a bounded density on $\mathbb{R}^d$. Given $A \subset \mathbb{R}_+ \times \mathbb{R}^d$, we denote $\Xi(A) \sim \text{Poi}(\Lambda(A))$ to be the number of points generated by Ξ in A. Furthermore, define

$$\hat{\Xi} \equiv \Xi - \Lambda,$$

to be the compensated $(\mathcal{F}_t)$-PPP for Ξ. It will be useful to refer to all the points generated by Ξ up to time t. We will denote this by the following process,

$$\Xi_t \equiv \Xi([0,t] \times \mathbb{R}^d).$$

We similarly define Λ_t and $\hat{\Xi}_t$. By (Ikeda and Watanabe, 2014, §III.3), $\hat{\Xi}_t$ is a $(\mathcal{F}_t)$-martingale in t with the previsible square function $\langle\hat{\Xi}\rangle_t = \Lambda_t$ and quadratic variation $[\hat{\Xi}]_t = \Xi_t$, i.e. the number of jumps of $\hat{\Xi}$ until time t.

Suppose $f : \mathbb{R}_+ \times \mathbb{R}^d \times \Omega \to \mathbb{R}$ is square integrable and $(\mathcal{F}_t)$-predicable and $f_t(x)$ is Borel-measurable. We denote the integral of f with respect to Ξ by,

$$A_t(f) \equiv \int_0^t \int f_s(x)\Xi(ds, dx) = \sum_{(t_i,x_i)\sim\Xi} 1(t_i \leq t) f_{t_i}(x_i).$$

If $A_t(f) < \infty$ a.s., then the compensated integral,

$$\hat{A}_t(f) \equiv \int_0^t \int f_s(x)\hat{\Xi}(ds, dx) = \int_0^t \int f_s(x)\Xi(ds, dx) - \int_0^t \int f_s(x)\Lambda(ds, dx),$$

is the stochastic integral of f with respect to $\hat{\Xi}$. By (Ikeda and Watanabe, 2014, §III.3), we know that $\hat{A}_t$ is a $(\mathcal{F}_t)$-martingale with the previsible square function,

$$\langle\hat{A}(f)\rangle_t = \int_0^t \int f_s(x)^2\Lambda(ds, dx),$$

and quadratic variation,

$$[\hat{A}(f)]_t = A_t(f) = \int_0^t \int f_s(x)^2\Xi(ds, dx).$$

Finally, it will be useful to define common spaces that will be used throughout this paper. We define,

$$\begin{aligned}
C_b^\infty &\equiv \{f : \mathbb{R} \to \mathbb{R} | f \text{ is bounded and smooth}\},\\
C_c^\infty &\equiv \{f : \mathbb{R} \to \mathbb{R} | f \text{ is smooth with compact support}\},\\
C_0 &\equiv \left\{f : \mathbb{R} \to \mathbb{R} | f \text{ is continuous}, \lim_{|x|\to\infty} f(x) = 0\right\},\\
M_F &\equiv \{\mu | \mu \text{ is a finite measure on } \mathbb{R}\}.
\end{aligned}$$

Given a metric space (S, d), define $C(S)$ and $D(S)$ as,

$$\begin{aligned}
C(S) &\equiv \{f : \mathbb{R}_+ \to S | f \text{ is continuous}\},\\
D(S) &\equiv \{f : \mathbb{R}_+ \to S | f \text{ is cádlág}\}.
\end{aligned}$$

We equip $C(S)$ with the topology of uniform convergence on compact sets, and $D(S)$ with the Skorokhod J_1 topology. In this paper we be interested in the case where S is either $\mathbb{R}$, C_0, or M_F.

2.2 Construction and SPDE characterization of X^ε

For $\varepsilon > 0$, let Ξ^ε be a PPP on $\mathbb{R}_+ \times \mathbb{R}$ with rate $\Lambda^\varepsilon(dt, dx) = \varepsilon^{-1} dt\psi(x)dx$, where ψ is a bounded, compactly supported density on $\mathbb{R}$. Further suppose that $\phi : \mathbb{R} \to \mathbb{R}_+$ is a Lipschitz function supported on $[-1, 1]$ satisfying $\|\phi\|_1 = 1$, and $\|\phi\|_\infty \leq 1$. We define ϕ_ε^x by

$$\phi_\varepsilon^x(y) = \sqrt{\varepsilon}\phi\left(\frac{y-x}{\sqrt{\varepsilon}}\right).$$

Remark 2.2.1. Note that the compactness of ψ is not necessary and is only used to simplify some of the analysis in the proof of Theorem 2.5.1. The theorem is still true for ψ bounded but requires more effort.

Let (t_i, x_i) be the points generated by Ξ^ε, and $\{X^i\}$ be independent SBM's with initial law $\phi_\varepsilon^{x_i}(z)dz$ in chronological order. By Theorem III.4.2 in (Perkins, 2002, §III.4) have for each X^i, there exists process u^i with sample paths in $C(C_0)$ such that for all $X_t^i(dx) = u_t^i(x)dx$. Suppose $\kappa_1 = 1$. We define κ_i inductively by,

$$\kappa_i = 1\left(\sum_{t_j < t_i} \kappa_j u_{t_i - t_j}^j(x_i) = 0\right). \tag{2.2}$$

So $\kappa_i = 1$ if and only if, for all $j < i$, $\kappa_j = 1$ implies SBM $X_{t_i}^j$ does not occupy space at x_i. We define M_F-valued processes X^ε and $\bar{X}^\varepsilon$ by,

$$X_t^\varepsilon \equiv \sum_{t_i \leq t} \kappa_i X_{t-t_i}^i, \tag{2.3}$$

and,

$$\bar{X}_t^\varepsilon \equiv \sum_{s_i \leq t} X_{t-t_i}^i, \tag{2.4}$$

respectively. Note that X^ε and $\bar{X}^\varepsilon$ are processes with sample paths in $D(M_F)$. Since $X^i_t(dx) = u^i_t(x)dx$, we immediately have $X^\varepsilon_t(dx) = u^\varepsilon_t(x)dx$ for the C_0-valued process u^ε defined by

$$u^\varepsilon_t \equiv \sum_{t_i \le t} \kappa_i u^i_{t-t_i}. . \tag{2.5}$$

Note that u^ε has sample paths in $D(C_0)$ and if

$$u_{t-}(x) \equiv \lim_{s \to t^-} u_s(x),$$

then (2.2), and (2.5) give us,

$$\kappa_i = 1(u_{t_i-}(x_i) = 0). \tag{2.6}$$

It will also be useful to define the natural filtration $\mathcal{F}^\varepsilon_t$ and $\bar{\mathcal{F}}^\varepsilon_t$ by,

$$\mathcal{F}^\varepsilon_t = \sigma(X^\varepsilon_s, \bar{X}^\varepsilon_s, \Xi^\varepsilon_s | s \le t).$$

respectively.

Again, by Theorem III.4 in (Perkins, 2002, §III.4) there are independent $(\mathcal{F}^\varepsilon_t)$-white noises W^i on $\mathbb{R}_+ \times, \mathbb{R}$, such that for all $\phi \in C^\infty_b$, X^i satisfies the SPDE,

$$\begin{aligned} X^i_t(\phi) &= \int_0^t X^i_s\left(\frac{\Delta\phi}{2}\right)ds + \int_0^t \int \phi(x)\sqrt{u^i_s(x)}W^i(ds,dx) + \langle \phi, \phi^{x_i}_\varepsilon \rangle \\ &\equiv L^i_t(\phi) + M^i_t(\phi) + A^i_t(\phi). \end{aligned} \tag{2.7}$$

Where $M^i(\phi)$ is a continuous $(\mathcal{F}^\varepsilon_t)$-martingale with $\langle M^i(\phi)\rangle_t = \int_0^t X^i_s(\phi^2)ds$.

By substituting (2.7) into (2.3) and (2.4), we get the following SPDE characterization of X^ε and $\bar{X}^\varepsilon$.

Theorem 2.2.2. *For all* $\phi \in C^\infty_b$,

(a) There is a continuous $(\mathcal{F}^\varepsilon_t)$*-martingale* $M^\varepsilon(\phi)$ *with* $\langle M^\varepsilon(\phi)\rangle_t = \int_0^t X^\varepsilon_s(\phi^2)ds$*, such that* X^ε *satisfies the SPDE,*

$$X^\varepsilon_t(\phi) = \int_0^t X^\varepsilon_s\left(\frac{\Delta\phi}{2}\right)ds + M^\varepsilon_t(\phi) + \int_0^t \int \langle \phi, \phi^x_\varepsilon \rangle 1(u^\varepsilon_{s-}(x) = 0)\Xi^\varepsilon(ds,dx) \tag{2.8}$$

$$\equiv L^\varepsilon_t(\phi) + M^\varepsilon_t(\phi) + A^\varepsilon_t(\phi). \tag{2.9}$$

(b) There is a continuous $(\mathcal{F}^\varepsilon_t)$-martingale $\bar{M}^\varepsilon(\phi)$ with $\langle \bar{M}^\varepsilon(\phi)\rangle_t = \int_0^t \bar{X}^\varepsilon_s(\phi^2)ds$, such that $\bar{X}^\varepsilon$ satisfies the SPDE,

$$\begin{aligned} \bar{X}^\varepsilon_t(\phi) = & \int_0^t \bar{X}^\varepsilon_s\left(\frac{\Delta\phi}{2}\right)ds + \bar{M}^\varepsilon_t(\phi) + \int_0^t \int \langle \phi, \phi^x_\varepsilon\rangle \Xi^\varepsilon(ds,dx) && (2.10)\\ \equiv & \ \bar{L}^\varepsilon_t(\phi) + \bar{M}^\varepsilon_t(\phi) + \bar{A}^\varepsilon_t(\phi). && (2.11) \end{aligned}$$

Proof. (a) Suppose (t_i, x_i) are the points generated by Ξ^ε. We begin by substituting (2.7) into (2.3).

$$\begin{aligned} X^\varepsilon_t(\phi) = & \sum_{t_i\le t} \kappa_i X^i_{t-t_i} \\ = & \sum_{t_i\le t} \kappa_i \left(\int_0^{t-t_i} X^i_s\left(\frac{\Delta\phi}{2}\right)ds + M^i_{t-t_i}(\phi) + \langle \phi, \phi^{x_i}_\varepsilon\rangle\right) \\ = & \sum_{t_i\le t} \left(\int_0^{t-t_i} \kappa_i X^i_s\left(\frac{\Delta\phi}{2}\right)ds + \kappa_i M^i_{t-t_i}(\phi) + \kappa_i\langle \phi, \phi^{x_i}_\varepsilon\rangle\right) \\ \equiv & \ L^\varepsilon_t(\phi) + M^\varepsilon_t(\phi) + A^\varepsilon_t(\phi). \qquad (2.12) \end{aligned}$$

It remains to show that $L^\varepsilon(\phi)$, $M^\varepsilon(\phi)$, and $A^\varepsilon(\phi)$, satisfy (2.8). We will deal each sum individually. To simplify $L^\varepsilon(\phi)$ we exchange the sum and integral.

$$\begin{aligned} L^\varepsilon_t(\phi) &= \sum_{t_i\le t}\int_0^{t-t_i} \kappa_i X^i_s\left(\frac{\Delta\phi}{2}\right)ds \\ &= \sum_{t_i\le t}\int_0^{t} 1(t_i\le s)\kappa_i X^i_{s-t_i}\left(\frac{\Delta\phi}{2}\right)ds \\ &= \int_0^{t}\sum_{t_i\le t} 1(t_i\le s)\kappa_i X^i_{s-t_i}\left(\frac{\Delta\phi}{2}\right)ds \\ &= \int_0^{t}\sum_{t_i\le s} \kappa_i X^i_{s-t_i}\left(\frac{\Delta\phi}{2}\right)ds \\ &= \int_0^{t} X^\varepsilon_s\left(\frac{\Delta\phi}{2}\right)ds. \qquad (2.13) \end{aligned}$$

Also note that by (2.6),

$$\begin{aligned} A_t^\varepsilon(\phi) &= \sum_{t_i \le t} \kappa_i \langle \phi, \phi_\varepsilon^x \rangle \\ &= \sum_{t_i \le t} 1(u_{t_i-}^\varepsilon(x_i) = 0) \langle \phi, \phi_\varepsilon^{x_i} \rangle \\ &= \int_0^t \int 1(u_{s-}^\varepsilon(x) = 0) \langle \phi, \phi_\varepsilon^x \rangle \Xi^\varepsilon(ds, dx). \end{aligned} \tag{2.14}$$

It remains to show that $M_t^\varepsilon(\phi) = \sum_{t_i \le t} \kappa_i M_{t-t_i}^i(\phi)$ satisfies the properties of the theorem:

(i) $M^\varepsilon(\phi)$ is continuous,

(ii) $M^\varepsilon(\phi)$ is a martingale,

(iii) $\langle M^\varepsilon(\phi) \rangle_t = \int_0^t X_s^\varepsilon(\phi^2) ds$.

We will deal with each one at a time.

(i) Note that $M^i(\phi)$ are continuous $(\mathcal{F}_t^\varepsilon)$-martingales starting from 0, and since $M^\varepsilon(\phi)$ is a sum of $M^i(\phi)$, we have $M^\varepsilon(\phi)$ must also be continuous.

(ii) Since $M^i(\phi)$ is a $(\mathcal{F}_t^\varepsilon)$-martingale, for $t \ge t_i$, $M_{t-t_i}^i(\phi)$ is $\mathcal{F}_{t-t_i}^\varepsilon$-measurable, and hence $\mathcal{F}_t^\varepsilon$-measurable since $\mathcal{F}_{t-t_i}^\varepsilon \subset \mathcal{F}_t^\varepsilon$. Also since Ξ_t^ε is $\mathcal{F}_t^\varepsilon$-measurable, we can conclude $M_t^\varepsilon(\phi) = \sum_{t_i \le t} \kappa_i M_{t-t_i}^i(\phi)$ is $\mathcal{F}_t^\varepsilon$-measurable.

To show the martingale property, note that we can decompose $M_t^\varepsilon(\phi)$ as,

$$\begin{aligned} M_t^\varepsilon(\phi) &= \sum_{t_i \le t} \kappa_i M_{t-t_i}^i(\phi) \\ &= \sum_{t_i \le t} \kappa_i \int_0^{t-t_i} \int \phi(x) \sqrt{u_s^i(x)} W^i(ds, dx) \\ &= \sum_{t_i \le t} \kappa_i \int_{t_i}^t \int \phi(x) \sqrt{u_{v-t_i}^i(x)} W^i(dv, dx). \end{aligned} \tag{2.15}$$

For $s \le t$, we separating (2.15) further into the time before s and after s.

$$\begin{aligned}M_t^\varepsilon(\phi) =&\sum_{t_i\le s}\kappa_i\int_{t_i}^s\int\phi(x)\sqrt{u^i_{v-t_i}(x)}W^i(dv,dx)\\&+\sum_{t_i\le s}\kappa_i\int_s^t\int\phi(x)\sqrt{u^i_{v-t_i}(x)}W^i(dv,dx)\\&+\sum_{s<t_i\le t}\kappa_i\int_{t_i}^t\int\phi(x)\sqrt{u^i_{v-t_i}(x)}W^i(dv,dx)\\=&M_s^\varepsilon(\phi)+\sum_{t_i\le s}\kappa_i\int_s^t\int\phi(x)\sqrt{u^i_{v-t_i}(x)}W^i(dv,dx)+\sum_{s<t_i\le t}\kappa_i M^i_{t-t_i}(\phi).\end{aligned} \tag{2.16}$$

We now take conditional expectation of the second and third terms in (2.16) with respect to $\mathcal{F}_s^\varepsilon$. Note that when $t_i \le s$, κ_i is $\mathcal{F}_s^\varepsilon$-measurable and for all $n \in \mathbb{N}$, $1(\Xi_s^\varepsilon = n)$ is $\mathcal{F}_s^\varepsilon$-measurable. This coupled with the fact that $\int_s^t\int\phi(x)\sqrt{u^i_{v-t_i}(x)}W^i(dv,dx)$ is a $(\mathcal{F}_t^\varepsilon)$-local martingale for $s \ge t$, gives us

$$\begin{aligned}&\mathbb{E}\left(\sum_{t_i\le s}\kappa_i\int_s^t\int\phi(x)\sqrt{u^i_{v-t_i}(x)}W^i(dv,dx)\Big|\mathcal{F}_s^\varepsilon\right)\\=&\ \mathbb{E}\left(\sum_{i\le n}1(\Xi_s^\varepsilon=n)\kappa_i\int_s^t\int\phi(x)\sqrt{u^i_{v-t_i}(x)}W^i(dv,dx)\Big|\mathcal{F}_s^\varepsilon\right)\\=&\ \sum_{i\le n}1(\Xi_s^\varepsilon=n)\kappa_i\mathbb{E}\left(\int_s^t\int\phi(x)\sqrt{u^i_{v-t_i}(x)}W^i(dv,dx)\Big|\mathcal{F}_s^\varepsilon\right)\\=&\ 0.\end{aligned} \tag{2.17}$$

To deal with $\sum_{s<t_i\le t}\kappa_i M^i_{t-t_i}(\phi)$, note that $\Xi_t^\varepsilon - \Xi_s^\varepsilon$ is independent of $\mathcal{F}_s^\varepsilon$, and since $s < t_i \le t$, we have $M^i_{t-t_i}(\phi) = \int_{t_i}^t\int\phi(x)\sqrt{u^i_{v-t_i}(x)}W^i(dv,dx)$ and κ_i are independent of $\mathcal{F}_s^\varepsilon$. So the third term of (2.16) is independent of $\mathcal{F}_s^\varepsilon$, and thus,

$$\mathbb{E}\left(\sum_{s<t_i\le t}\kappa_i M^i_{t-t_i}(\phi)\Big|\mathcal{F}_s^\varepsilon\right) = \mathbb{E}\left(\sum_{s<t_i\le t}\kappa_i M^i_{t-t_i}(\phi)\right). \tag{2.18}$$

Note that $M^i(\phi)$ is independent of κ_i, so by conditioning on κ_i we get

$$\mathbb{E}\left(\kappa_i M^i_{t-t_i}(\phi)\right) = \mathbb{E}\left(\kappa_i\right)\mathbb{E}(M^i_{t-t_i}(\phi)) = 0. \tag{2.19}$$

Since $\sum_{s<t_i\le t}\kappa_i M^i_{t-t_i}(\phi)$ is the sum of $\Xi^\varepsilon_t - \Xi^\varepsilon_s$ independent random variables with mean 0, Wald's equation gives us,

$$\mathbb{E}\left(\sum_{s<t_i\le t}\kappa_i M^i_{t-t_i}(\phi)\right) = \mathbb{E}\left(\Xi^\varepsilon_t - \Xi^\varepsilon_s\right)\mathbb{E}\left(\kappa_i M^i_{t-t_i}(\phi)\right) = \frac{t-s}{\varepsilon}\cdot 0 = 0. \tag{2.20}$$

Combining (2.16), (2.17), (2.18), and (2.20), we get,

$$\mathbb{E}(M^\varepsilon_t(\phi)|\mathcal{F}^\varepsilon_s) = M^\varepsilon_s(\phi).$$

Thus $M^\varepsilon(\phi)$ is a $(\mathcal{F}^\varepsilon_t)$-martingale.

(iii) We define $N^\varepsilon_t(\phi) \equiv M^\varepsilon_t(\phi)^2 - \int_0^t X^\varepsilon_u(\phi^2)du$. In order to show $\langle M^\varepsilon(\phi)\rangle_t = \int_0^t X^\varepsilon_u(\phi^2)du$, we need $N^\varepsilon(\phi)$ to be a $(\mathcal{F}^\varepsilon_t)$-martingale. Since $M^\varepsilon_t(\phi)$ is a $\mathcal{F}^\varepsilon_t$-measurable, so is $N^\varepsilon_t(\phi)$. Before we prove the martingale property, it will be useful to define $N^i(\phi)$ to be the square $(\mathcal{F}^\varepsilon_t)$-martingale for $M^i(\phi)$. By definition of $\langle M^i(\phi)\rangle$,

$$N^i_t(\phi) \equiv M^i_t(\phi)^2 - \langle M^i(\phi)\rangle_t = M^i_t(\phi)^2 - \int_0^t X^i_u(\phi^2)du.$$

Note that for $t \ge t_i$,

$$\langle M^i(\phi)\rangle_{t-t_i} = \int_0^{t-t_i} X^i_u(\phi^2)du = \int_0^t 1(t_i \le u)X^i_{u-t_i}(\phi^2)du. \tag{2.21}$$

We simplify $\int_0^t X^\varepsilon_u(\phi^2)du$ by using (2.21),

$$\begin{aligned}\int_0^t X^\varepsilon_u(\phi)du &= \int_0^t \sum_{t_i\le u}\kappa_i X^i_{u-t_i}(\phi^2)du \\ &= \int_0^t \sum_{t_i\le t}\kappa_i 1(t_i\le u)X^i_{u-t_i}(\phi^2)du \\ &= \sum_{t_i\le t}\kappa_i \int_0^t 1(t_i\le u)X^i_{u-t_i}(\phi^2)du \\ &= \sum_{t_i\le t}\kappa_i\langle M^i(\phi)\rangle_{t-t_i}. \end{aligned} \tag{2.22}$$

We can now use (2.22) to simplify $N^\varepsilon(\phi)_t$.

$$\begin{aligned}N_t^\varepsilon(\phi) &= M_t^\varepsilon(\phi)^2 - \int_0^t X_u^\varepsilon(\phi^2)du \\ &= \left(\sum_{t_i \le t} \kappa_i M_{t-t_i}^i(\phi)\right)^2 - \sum_{t_i \le t} \kappa_i \langle M^i(\phi)\rangle_{t-t_i} \\ &= \sum_{t_i \le t} \kappa_i \left(M_{t-t_i}^i(\phi)^2 - \langle M^i(\phi)\rangle_{t-t_i}\right) + 2 \sum_{t_j < t_i \le t} \kappa_j \kappa_i M_{t-t_j}^j(\phi) M_{t-t_i}^i(\phi) \\ &= \sum_{t_i \le t} \kappa_i N_{t-t_i}^i(\phi) + 2 \sum_{t_j < t_i \le t} \kappa_j \kappa_i M_{t-t_j}^j(\phi) M_{t-t_i}^i(\phi). \end{aligned} \tag{2.23}$$

For $s < t$, we further decompose (2.23) into the clusters born before and after time s.

$$\begin{aligned}N_t^\varepsilon(\phi) = &\sum_{t_i \le s} \kappa_i N_{t-t_i}^i(\phi) + 2 \sum_{t_j < t_i \le s} \kappa_j \kappa_i M_{t-t_j}^j(\phi) M_{t-t_i}^i(\phi) \\ &+ \sum_{s < t_i \le t} \kappa_i N_{t-t_i}^i(\phi) + 2 \sum_{t_j < t_i, s < t_i} \kappa_j \kappa_i M_{t-t_j}^j(\phi) M_{t-t_i}^i(\phi). \end{aligned} \tag{2.24}$$

Similar to the proof of (ii), when $t_i \le s$, κ_i is $\mathcal{F}_s^\varepsilon$-measurable and for all $n \in \mathbb{N}$, $1(\Xi_s^\varepsilon = n)$ is $\mathcal{F}_s^\varepsilon$-measurable. Also, $N_{t-t_i}^i$ and $M_{t-t_j}^j(\phi) M_{t-t_i}^i(\phi)$ are a $(\mathcal{F}_t^\varepsilon)$-martingales for $t \ge t_i$, since $M^j(\phi)$ is independent of $M^i(\phi)$. So the conditional expectation of the first term in (2.24) becomes,

$$\begin{aligned}\mathbb{E}\left(\sum_{t_i \le s} \kappa_i N_{t-t_i}^i(\phi) \middle| \mathcal{F}_s^\varepsilon\right) &= \sum_{n \ge 0} \kappa_i 1(\Xi_s^\varepsilon = n) \sum_{i \le n} \mathbb{E}(N_{t-t_i}^i(\phi) | \mathcal{F}_s^\varepsilon) \\ &= \sum_{n \ge 0} \kappa_i 1(\Xi_s^\varepsilon = n) \sum_{i \le n} \mathbb{N}_{s-t_i}^i(\phi) \\ &= \sum_{t_i \le s} \kappa_i N_{s-t_i}^i(\phi). \end{aligned} \tag{2.25}$$

Similarly the conditional expectation of the second term becomes,

$$\mathbb{E}\left(2 \sum_{t_j < t_i \le s} \kappa_i \kappa_j M_{t-t_j}^j(\phi) M_{t-t_i}^i(\phi) \middle| \mathcal{F}_s^\varepsilon\right) = 2 \sum_{t_j < t_i \le s} \kappa_i \kappa_j M_{s-t_j}^j(\phi) M_{s-t_i}^i(\phi). \tag{2.26}$$

For the other two terms of (2.24), again, as in (ii), note that $\Xi^\varepsilon_t - \Xi^\varepsilon_s$ is independent of $\mathcal{F}^\varepsilon_s$, and since $s < t_i \le t$, $N^\varepsilon_{t-t_i}$ and $M^i_{t-t_j}$ are independent of $\mathcal{F}^\varepsilon_s$, we have

$$\mathbb{E}\left(\sum_{s<t_i\le t}\kappa_i N^i_{t-t_i}(\phi) + 2\sum_{t_j<t_i, s<t_i}\kappa_i\kappa_j M^j_{t-t_j}(\phi)M^i_{t-t_i}(\phi)\Big|\mathcal{F}^\varepsilon_s\right)$$

$$= \mathbb{E}\left(\sum_{s<t_i\le t}\kappa_i N^i_{t-t_i}(\phi) + 2\sum_{t_j<t_i, s<t_i}\kappa_i\kappa_j M^j_{t-t_j}(\phi)M^i_{t-t_i}(\phi)\right) \quad (2.27)$$

$$= 0. \quad (2.28)$$

In the last line we used the fact that each term in the random sums in (2.27) has mean 0 along with Wald's equation, as previously done in (2.19). Therefore combining (2.24), (2.25), (2.26), and (2.28), we have shown

$$\mathbb{E}(N^\varepsilon_t(\phi)|\mathcal{F}^\varepsilon_s) = \sum_{t_i\le s}\kappa_i N^i_{s-t_i}(\phi) + 2\sum_{t_i<t_j\le s}\kappa_i\kappa_j M^i_{s-t_i}(\phi)M^j_{s-t_j}(\phi) = N^\varepsilon_s(\phi).$$

Thus $N^\varepsilon(\phi)$ is a $(\mathcal{F}^\varepsilon_t)$-martingale and $\langle M^\varepsilon(\phi)\rangle_t = \int_0^t X_u(\phi^2)du$.

(b) The proof is identical to (a) but with $\kappa_i = 1$ for all i.

□

We end this section by noting the trivial but very useful observation.

Proposition 2.2.3. *For all non-negative measurable* ϕ,

(a) $X^\varepsilon(\phi) \le \bar{X}^\varepsilon(\phi)$.

(b) $A^\varepsilon(\phi) \le \bar{A}^\varepsilon(\phi)$.

Proof. (a) This is immediate from (2.3), and (2.4).

(b) This is immediate from the definition of A^ε and $\bar{A}^\varepsilon$.

□

2.3 Tightness

Suppose $\varepsilon_n \to 0$ as $n \to \infty$. The goal for this section is to establish the existence of continuous processes X, and A as weak limits points of $\{X^{\varepsilon_n}\}$ and $\{A^{\varepsilon_n}\}$. We will do this by showing the two families of processes are C-relatively compact in $D(M_F)$.

Definition 2.3.1. *Let (S,d) be a Polish space. A collection of processes $\{X^\alpha\}_{\alpha\in I}$ with paths in $D(S)$ is C-**relatively compact** in $D(S)$ iff it is relatively compact in $D(S)$ and all weak limit points are continuous a.s.*

Definition 2.3.2. *$D_0 \subset C_b^\infty(\mathbb{R})$ is **separating** iff for all $\mu_1, \mu_2 \in M_F$, with $\mu_1(\phi) = \mu_2(\phi)$ for all $\phi \in D_0$, then $\mu_1 = \mu_2$.*

Theorem 2.3.3 (Jakubowski)**.** *(Perkins, 2002, §II.4) Let $D_0 \subset C_b^\infty$ be separating. A sequence of processes X^n in $D(M_F)$ is C-relatively compact in $D(M_F)$ iff*

(i) $\forall \phi \in D_0$, the sequence $\{X^n(\phi)\}_n$ is C-relatively compact in $D(\mathbb{R})$.

(ii) The compact containment condition holds, i.e, $\forall \eta, T > 0$ there is a compact set $K_{\eta,T} \subset \mathbb{R}$ such that

$$\sup_n \mathbb{P}\left(\sup_{t\le T} X_t^n(K_{\eta,T}) > \eta\right) \le \eta.$$

Theorem 2.3.4. *Let $\varepsilon_n \to 0$, as $n \to \infty$, then*

(a) X^{ε_n} is C-relatively compact in $D(M_F)$.

(b) A^{ε_n} is C-relatively compact in $D(M_F)$.

Remark 2.3.5. An immediate consequence of Theorem 2.3.4 is that there exists a common subsequence $\varepsilon_n \to 0$, and continuous M_F-valued processes X and A such that $(X^{\varepsilon_n}, A^{\varepsilon_n})$ converges weakly to (X, A).

To prove Theorem 2.3.4, we will invoke Jakubowski's theorem using the separating class $D_0 = C_c^\infty$, smooths functions with compact support. We will prove both condition's (i) and (ii) of Theorem 2.3.3 in the next 2 subsections.

2.3.1 Proof of Theorem 2.3.4: C-relative compactness of $\{X^{\varepsilon_n}(\phi)\}$, and $\{A^{\varepsilon_n}(\phi)\}$

Our goal in this section is to show that for all $\phi \in C_c^\infty$, $\{X^\varepsilon(\phi)\}$ and $\{A^\varepsilon(\phi)\}$ are C-relative compactness in $D(\mathbb{R})$. Our main weapons to tackle this problem will the following theorems:

Theorem 2.3.6. *Let $\{Y^n\}$ be a a family of stochastic processes with sample paths in $C(\mathbb{R})$. Suppose for all $T > 0$, there exists $\alpha, \beta, K > 0$ such that for all $0 < s, t < T$ and n,*

$$\mathbb{E}\left(|Y_t^n - Y_s^n|^\alpha\right) \leq K|t-s|^{1+\beta}.$$

Then $\{Y^n\}$ is C-relatively compact in $C(\mathbb{R})$.

Proof. This is a well known result, and can be deduced from Corollary 3.8.10 and 3.10.3 in (Ethier and Kurtz, 2009). ☐

Theorem 2.3.7 (Slutsky's)**.** *Let (S, d) be a metric space, and (X_n, X) be random elements of $S \times S$. If $Y_n \to Y$ and $d(X_n, Y_n) \to 0$ weakly, then $X_n \to Y$ weakly.*

Proof. This is Theorem 3.1 in (Billingsley, 2013, §1). ☐

Recall that by (2.9) in Theorem 2.2.2,

$$X_t^\varepsilon(\phi) = L_t^\varepsilon(\phi) + M_t^\varepsilon(\phi) + A_t^\varepsilon(\phi).$$

We begin by simplifying $A_t^\varepsilon(\phi)$.

$$\begin{aligned}
& A_t^\varepsilon(\phi) \\
= \; & \int_0^t \int \langle \phi, \phi_\varepsilon^x \rangle 1(u_{s-}^\varepsilon(x) = 0)\Xi^\varepsilon(ds, dx) \\
= \; & \int_0^t \int \langle \phi, \phi_\varepsilon^x \rangle 1(u_{s-}^\varepsilon(x) = 0)\Lambda^\varepsilon(ds, dx) + \int_0^t \int \langle \phi, \phi_\varepsilon^x \rangle 1(u_{s-}^\varepsilon(x) = 0)\hat{\Xi}^\varepsilon(ds, dx) \\
= \; & \int_0^t \int \langle \phi, \phi_\varepsilon^x \rangle \frac{1}{\varepsilon}\psi(x) 1(u_{s-}^\varepsilon(x) = 0)dxds + \int_0^t \int \langle \phi, \phi_\varepsilon^x \rangle 1(u_{s-}^\varepsilon(x) = 0)\hat{\Xi}^\varepsilon(ds, dx) \\
\equiv \; & I_t^\varepsilon(\phi) + \hat{A}_t^\varepsilon(\phi).
\end{aligned} \tag{2.29}$$

Where $I^\varepsilon(\phi)$ is continuous process $\mathbb{R}$-valued process, and $\hat{A}^\varepsilon(\phi)$ is a $(\mathcal{F}_t^\varepsilon)$-martingale with jumps. Therefore X_t^ε can be decomposed as

$$X_t^\varepsilon(\phi) = L_t^\varepsilon(\phi) + M_t^\varepsilon(\phi) + I_t^\varepsilon(\phi) + \hat{A}_t^\varepsilon(\phi). \tag{2.30}$$

To show the C-relative compactness of $\{X^\varepsilon(\phi)\}$, we will handle each term of (2.30) individually. Since $L^\varepsilon(\phi)$, $A^\varepsilon(\phi)$, and $I^\varepsilon(\phi)$ all have sample paths in $C(\mathbb{R})$, Theorem 2.3.6 tells us that we want to find estimate on the moments of the increments. Even though $\hat{A}^\varepsilon(\phi)$ contains jumps, we will show that it converges to 0 in probability as $\varepsilon \to 0$ thus allowing us to use Slutsky's theorem. We proceed to this in the next few lemmas.

Before we can bound the increments of $L^\varepsilon(\phi)$, $M^\varepsilon(\phi)$, and $I^\varepsilon(\phi)$, the following estimate on the total mass of $\bar{X}^\varepsilon$ will be useful.

Lemma 2.3.8. *For all $T > 0$, and $p = 2^n$, $n \geq 0$, there are constants $C(p,T)$ such that for all $0 \leq t \leq T$,*

(a) $\mathbb{E}(\bar{X}_t^\varepsilon(1)) = t$,

(b) $\mathbb{E}\left(\bar{X}_t^\varepsilon(1)^p\right) \leq C(p,T)t^p + \delta_{p,T}(\varepsilon)$,

Where $\delta_{p,T}(\varepsilon) \to 0$ as $\varepsilon \to 0$.

Proof. (a) By letting $\phi = 1$ in (2.10), we get the that $\bar{X}^\varepsilon(1)$ solves the SDE

$$\begin{aligned}
\bar{X}_t^\varepsilon(1) &= \bar{M}_t^\varepsilon(1) + \bar{A}_t^\varepsilon(1) \\
&= \bar{M}_t^\varepsilon(1) + \int_0^t \int \langle 1, \phi_\varepsilon^x \rangle \Xi^\varepsilon(ds, dx) \\
&= \bar{M}_t^\varepsilon(1) + \int_0^t \int \varepsilon \hat{\Xi}^\varepsilon(ds, dx) + \int_0^t \int \varepsilon \Lambda^\varepsilon(ds, dx) \\
&= \bar{M}_t^\varepsilon(1) + \varepsilon \hat{\Xi}_t^\varepsilon + \int_0^t \int \varepsilon \frac{1}{\varepsilon} \psi(x) dx ds \\
&= \bar{M}_t^\varepsilon(1) + \varepsilon \hat{\Xi}_t^\varepsilon + t.
\end{aligned} \tag{2.31}$$

Since $\bar{M}^\varepsilon$ and $\hat{\Xi}^\varepsilon$ are $(\mathcal{F}_t^\varepsilon)$-martingales, by taking expectations in (2.31) immediately gives us,

$$\mathbb{E}\left(\bar{X}_t(1)\right) = t.$$

(b) Let $T > 0$. We now proceed by induction on n for $p = 2^n$. In the case where $p = 1$ was shown in part (a). Suppose that there is a $C(p,T)$, $\delta_{p,T}(\varepsilon) \to 0$ for $\varepsilon \to 0$ such that,

$$\mathbb{E}\left(\bar{X}_t(1)^p\right) \leq C(p,T)t^p + \delta_{p,T}(\varepsilon). \tag{2.32}$$

Lets now estimate the $2p$-th moment. By applying Jensen's inequality for sums to (2.31) we get,

$$\begin{aligned}
\mathbb{E}\left(\bar{X}_t^\varepsilon(1)^{2p}\right) &= \mathbb{E}\left((\bar{M}_t^\varepsilon(1) + \varepsilon\hat{\Xi}_t^\varepsilon + t)^{2p}\right) \\
&\leq 3^{2p-1}\mathbb{E}\left(\bar{M}_t^\varepsilon(1)^{2p} + \varepsilon^{2p}(\hat{\Xi}_t^\varepsilon)^{2p} + t^{2p}\right) \\
&= 3^{2p-1}\left[\mathbb{E}\left(\bar{M}_t^\varepsilon(1)^{2p}\right) + \varepsilon^{2p}\mathbb{E}\left((\hat{\Xi}_t^\varepsilon)^{2p}\right) + t^{2p}\right] \\
&= 3^{2p-1}C_p\left[\mathbb{E}\left([\bar{M}^\varepsilon(1)]_t^p\right) + \varepsilon^{2p}\mathbb{E}\left([\hat{\Xi}^\varepsilon]_t^p + 1\right) + t^{2p}\right] \\
&= 3^{2p-1}C_p\left[\mathbb{E}\left(\left(\int_0^t \bar{X}_s^\varepsilon(1)ds\right)^p\right) + \varepsilon^{2p}\mathbb{E}\left((\Xi_t^\varepsilon)^p\right) + \varepsilon^{2p} + t^{2p}\right].
\end{aligned} \tag{2.33}$$

The second last line is true for some $C_p > 0$ by the continuous version Burkholder-Davis-Gundy inequality for previsible jump martingales as stated in (Perkins, 2002, §II.4), and by noting $\hat{\Xi}^\varepsilon$ has jumps of size 1. The last line is true because $[\bar{M}^\varepsilon(1)]_t = \langle\bar{M}^\varepsilon(1)\rangle_t = \int_0^t \bar{X}_s^\varepsilon(1)ds$ and $\langle\hat{\Xi}^\varepsilon\rangle_t = \Xi_t^\varepsilon$. We will handle the two expectation terms in (2.33) individually.

We bound the integral term using Jensen's inequality for x^p and our induction hypothesis (2.32),

$$\begin{aligned}
\mathbb{E}\left(\left(\int_0^t \bar{X}_s^\varepsilon(1)ds\right)^p\right) &\leq t^{p-1}\mathbb{E}\left(\int_0^t \bar{X}_s^\varepsilon(1)^p ds\right) \\
&= t^{p-1}\int_0^t \mathbb{E}\left(\bar{X}_s^\varepsilon(1)^p\right) ds \\
&= t^{p-1}\int_0^t C(p,T)s^p + \delta_{p,T}(\varepsilon)ds \\
&= \frac{C(p,T)}{p+1}t^{2p} + t^p\delta_{p,T}(\varepsilon) \\
&\leq \frac{C(p,T)}{p+1}t^{2p} + T^p\delta_{p,T}(\varepsilon).
\end{aligned} \tag{2.34}$$

For the Ξ^ε term in (2.33), we note that $\Xi_t^\varepsilon \sim \text{Poi}\left(\frac{t}{\varepsilon}\right)$. It is an elementary fact that the m-th moment of a Poisson random variable with mean λ is a polynomial in λ of degree m with zero constant term. Which implies $\mathbb{E}\left((\Xi_t^\varepsilon)^p\right)$ is a polynomial in $\frac{t}{\varepsilon}$ of degree p and thus,

$$\lim_{\varepsilon \to 0} \varepsilon^{2p} \mathbb{E}\left((\Xi_t^\varepsilon)^p\right) = 0. \tag{2.35}$$

Therefore combining (2.33), (2.34), and (2.35), we can conclude,

$$\begin{aligned}
& \mathbb{E}\left(\bar{X}_t^\varepsilon(1)^{2p}\right) \\
\leq\ & 3^{2p-1} C_p \left[\frac{C(p,T)}{p+1} t^{2p} + T^p \delta_{p,T}(\varepsilon) + \varepsilon^{2p} \mathbb{E}\left((\Xi_t^\varepsilon)^p\right) + \varepsilon^{2p} + t^{2p}\right] \\
=\ & 3^{2p-1} C_p \left(\frac{C(p,T)}{p+1} + 1\right) t^{2p} + 3^{2p-1} C_p \left(T^p \delta_{p,T}(\varepsilon) + \varepsilon^{2p} \mathbb{E}\left((\Xi_t^\varepsilon)^p\right) + \varepsilon^{2p}\right) \\
\equiv\ & C(2p,T) t^{2p} + \delta_{2p,T}(\varepsilon).
\end{aligned}$$

Where $\delta_{2p,T}(\varepsilon) \to 0$ as $\varepsilon \to 0$.

□

We will now use Lemma 2.3.8 to estimate the increment moments of $L^{\varepsilon_n}(\phi)$, $M^{\varepsilon_n}(\phi)$ and $I^{\varepsilon_n}(\phi)$.

Lemma 2.3.9. *Let $\varepsilon_n \to 0$ as $n \to \infty$, $T > 0$, $\phi \in C_c^\infty$, and $p = 2^m$ for $m \geq 0$. There exists $K_L(p,T,\phi), K_M(p,T,\phi)$ such that for all $0 < s < t \leq T$,*

(a) $\mathbb{E}\left(|L_t^{\varepsilon_n}(\phi) - L_s^{\varepsilon_n}(\phi)|^p\right) \leq K_L(p,T,\phi)|t-s|^p$

(b) $\mathbb{E}\left(|M_t^{\varepsilon_n}(\phi) - M_s^{\varepsilon_n}(\phi)|^{2p}\right) \leq K_M(p,T,\phi)|t-s|^p$

(c) $\mathbb{E}\left(|I_t^{\varepsilon_n}(\phi) - I_s^{\varepsilon_n}(\phi)|^p\right) \leq \|\phi\|_\infty^p |t-s|^p$

Proof. (a) We use the definition of $L^\varepsilon(\phi)$ and applying Jensen's inequality for $|x|^p$ to get,

$$\begin{aligned}
&\mathbb{E}\left(|L_t^{\varepsilon_n}(\phi) - L_s^{\varepsilon_n}(\phi)|^p\right) \\
= \ & \mathbb{E}\left(\left|\int_s^t X_u^{\varepsilon_n}\left(\frac{\Delta\phi}{2}\right) du\right|^p\right) \\
\leq \ & \left\|\frac{\Delta\phi}{2}\right\|_\infty^p \mathbb{E}\left(\left(\int_s^t \bar{X}_u^{\varepsilon_n}(1) du\right)^p\right) \\
\leq \ & \left\|\frac{\Delta\phi}{2}\right\|_\infty^p |t-s|^{p-1}\mathbb{E}\left(\int_s^t \bar{X}_u^{\varepsilon_n}(1)^p du\right) \\
= \ & \left\|\frac{\Delta\phi}{2}\right\|_\infty^p |t-s|^{p-1} \int_s^t \mathbb{E}\left(\bar{X}_u^{\varepsilon_n}(1)^p\right) du.
\end{aligned}$$

We can esimate the integrand using Lemma 2.3.8.

$$\begin{aligned}
&\mathbb{E}\left(|L_t^{\varepsilon_n}(\phi) - L_s^{\varepsilon_n}(\phi)|^p\right) \\
\leq \ & \left\|\frac{\Delta\phi}{2}\right\|_\infty^p |t-s|^{p-1} \int_s^t C(p,T)u^p + \delta_{p,T}(\varepsilon_n) du \\
= \ & \left\|\frac{\Delta\phi}{2}\right\|_\infty^p |t-s|^{p-1} \left(\frac{C(p,T)}{p+1}|t-s|^{p+1} + \delta_{p,T}(\varepsilon_n)|t-s|\right) \\
\leq \ & \left\|\frac{\Delta\phi}{2}\right\|_\infty^p \left(\frac{C(p,T)}{p+1}T^p + \delta_{p,T}(\varepsilon_n)\right)|t-s|^p \\
\leq \ & \left\|\frac{\Delta\phi}{2}\right\|_\infty^p \left(\frac{C(p,T)}{p+1}T^p + D_{p,T}\right)|t-s|^p.
\end{aligned}$$

Since $\delta_{p,T}(\varepsilon_n) \to 0$ as $n \to \infty$, $\delta_{p,T}(\varepsilon_n)$ is bounded by some $D_{p,T}$ uniformly in n. Therefore we have a constant $K_L(p,T,\phi)$ such that for all n,

$$\mathbb{E}\left(|L_t^{\varepsilon_n}(\phi) - L_s^{\varepsilon_n}(\phi)|^p\right) \leq K_L(p,T,\phi)|t-s|^p.$$

(b) Note that by the definition of $M^\varepsilon(\phi)$ in (2.8), $N_t^\varepsilon(\phi) \equiv M_t^\varepsilon(\phi) - M_s^\varepsilon(\phi)$ is a continuous $(\mathcal{F}_t^\varepsilon)$-martingale in t for $t \geq s$ with square function,

$$\langle N^\varepsilon(\phi)\rangle_t = \int_s^t X_u^\varepsilon(\phi^2) du.$$

Let's bound the p-th moments by applying Jensen's inequality for $|x|^{2p}$, and the

Burkholder-Davis-Gundy inequality.

$$\begin{aligned}
& \mathbb{E}\left(|M_t^{\varepsilon_n}(\phi) - M_s^{\varepsilon_n}(\phi)|^{2p}\right) \\
= \ & \mathbb{E}\left(|N_t^{\varepsilon_n}(\phi)|^{2p}\right) \\
\leq \ & C_p \mathbb{E}\left(\langle N^{\varepsilon_n}(\phi)\rangle_t^p\right) \\
= \ & C_p \mathbb{E}\left(\left(\int_s^t X_u^{\varepsilon_n}(\phi^2) du\right)^p\right) \\
\leq \ & C_p \|\phi\|_\infty^{2p} \mathbb{E}\left(\left(\int_s^t \bar{X}_u^{\varepsilon_n}(1) du\right)^p\right) \\
\leq \ & C_p \|\phi\|_\infty^{2p} |t-s|^{p-1} \mathbb{E}\left(\int_s^t \bar{X}_u^{\varepsilon_n}(1)^p du\right) \\
= \ & C_p \|\phi\|_\infty^{2p} |t-s|^{p-1} \int_s^t \mathbb{E}\left(\bar{X}_u^{\varepsilon_n}(1)^p\right) du.
\end{aligned}$$

Again, we apply Lemma 2.3.8 to get,

$$\begin{aligned}
& \mathbb{E}\left(|M_t^{\varepsilon_n}(\phi) - M_s^{\varepsilon_n}(\phi)|^{2p}\right) \\
\leq \ & C_p \|\phi\|_\infty^{2p} |t-s|^{p-1} \int_s^t C(p,T)\, u^p + \delta_{p,T}(\varepsilon_n) du \\
\leq \ & C_p \|\phi\|_\infty^{2p} |t-s|^{p-1} \left(\frac{C(p,T)}{p+1}|t-s|^{p+1} + D_{p,T}|t-s|\right) \\
= \ & C_p \|\phi\|_\infty^{2p} \left(\frac{C(p,T)}{p+1} T^p + D_{p,T}\right) |t-s|^p.
\end{aligned}$$

Just as in part (a), we assumed $\delta_{p,T}(\varepsilon_n)$ is bounded by $D_{p,T}$. Therefore there is a $K_M(p,T,\phi)$ such that for all n,

$$\mathbb{E}\left(|M_t^{\varepsilon_n}(\phi) - M_s^{\varepsilon_n}(\phi)|^{2p}\right) \leq K_M(p,T,\phi)|t-s|^p.$$

(c) We proceed directly by applying Hölder's inequality,

$$\begin{aligned}
\mathbb{E}\left(|I_t^{\varepsilon_n}(\phi) - I_s^{\varepsilon_n}(\phi)|^p\right) &= \mathbb{E}\left(\left|\int_s^t \int \langle \phi, \phi_{\varepsilon_n}^x \rangle \frac{1}{\varepsilon_n} \psi(x) 1(u_{s-}^{\varepsilon_n}(x) = 0) dx ds\right|^p\right) \\
&\leq \left|\int_s^t \int \|\phi\|_\infty \|\phi_{\varepsilon_n}^x\|_1 \frac{1}{\varepsilon_n} \psi(x) dx ds\right|^p \\
&= \|\phi\|_\infty^p |t-s|^p.
\end{aligned}$$

□

Lemma 2.3.10. *For all $T > 0$, $\phi \in C_b^\infty$. $\hat{A}_t^\varepsilon(\phi)$ converges to 0 uniformly on $[0, T]$ in L^2 as $\varepsilon \to 0$.*

Proof. Since $\hat{A}^{\varepsilon_n}(\phi)$ is a $(\mathcal{F}_t^{\varepsilon_n})$-martingale, Doob's strong L^2 inequality tells us,

$$\begin{aligned}
\mathbb{E}\left(\sup_{t\leq T} \hat{A}_t^\varepsilon(\phi)^2\right) &\leq 4\mathbb{E}\left(\hat{A}_T^\varepsilon(\phi)^2\right) \\
&= 4\mathbb{E}\left(\int_0^T \int \langle \phi, \phi_\varepsilon^x\rangle^2 1(u_{s-}^\varepsilon(x) = 0)\Xi^\varepsilon(ds, dx)\right) \\
&= 4\int_0^T \int \langle \phi, \phi_\varepsilon^x\rangle^2 1(u_{s-}^\varepsilon(x) = 0)\Lambda^\varepsilon(ds, dx) \\
&\leq 4\int_0^T \int \|\phi\|_\infty^2 \|\phi_\varepsilon^x\|_1^2 \frac{1}{\varepsilon}\psi(x) dx ds \\
&= 4T\|\phi\|_\infty^2 \varepsilon \\
&\to 0,
\end{aligned}$$

as $\varepsilon \to 0$. Therefore $A^{\varepsilon_n}(\phi)$ converges to 0 uniformly on $[0, T]$ in L^2. □

With Lemma 2.3.9 and 2.3.10 in hand, we are now ready to show $\{X^{\varepsilon_n}(\phi)\}$ and $\{A^{\varepsilon_n}(\phi)\}$ are C-relatively compact in $D(\mathbb{R})$.

Proposition 2.3.11. *Suppose $\varepsilon_n \to 0$, for all $\phi \in C_c^\infty(\mathbb{R})$, then*

(a) $\{X^{\varepsilon_n}(\phi)\}$ is C-relatively compact in $D(\mathbb{R})$,

(b) $\{A^{\varepsilon_n}(\phi)\}$ is C-relatively compact in $D(\mathbb{R})$.

Proof. (a) Let $\phi \in C_b$, and $T > 0$. Since $X^{\varepsilon_n}(\phi)$ satisfies (2.30),

$$\begin{aligned}
X_t^{\varepsilon_n}(\phi) &= L_t^{\varepsilon_n}(\phi) + M_t^{\varepsilon_n}(\phi) + I_t^{\varepsilon_n}(\phi) + \hat{A}_t^{\varepsilon_n}(\phi) \\
&\equiv Y_t^{\varepsilon_n}(\phi) + \hat{A}_t^{\varepsilon_n}(\phi).
\end{aligned}$$

Where $Y^{\varepsilon_n}(\phi)$ is a stochastic process with sample paths in $C(\mathbb{R})$, and $\hat{A}^{\varepsilon_n}(\phi)$ is a $(\mathcal{F}_t^{\varepsilon_n})$-martingale with jumps. We will first show that $Y^{\varepsilon_n}(\phi)$ is C-relatively compact.

For $0 < s < t \leq T$, we proceed by bounding $\mathbb{E}|Y_t^{\varepsilon_n} - Y_s^{\varepsilon_n}|^{2p}$, by applying Jensen's inequality to $|x|^{2p}$ and Lemma 2.3.9.

$$\begin{aligned}
& \mathbb{E}|Y_t^{\varepsilon_n} - Y_s^{\varepsilon_n}|^{2p} \\
= \; & \mathbb{E}\left|(L_t^{\varepsilon_n}(\phi) - L_s^{\varepsilon_n}(\phi)) + (M_t^{\varepsilon_n}(\phi) - M_s^{\varepsilon_n}(\phi)) + (I_t^{\varepsilon_n}(\phi) - I_s^{\varepsilon_n}(\phi))\right|^{2p} \\
\leq \; & 3^{2p-1}\left(\mathbb{E}\left|L_t^{\varepsilon_n}(\phi) - L_s^{\varepsilon_n}(\phi)\right|^{2p} + \mathbb{E}\left|M_t^{\varepsilon_n}(\phi) - M_s^{\varepsilon_n}(\phi)\right|^{2p} + \mathbb{E}\left|I_t^{\varepsilon_n}(\phi) - I_s^{\varepsilon_n}(\phi)\right|^{2p}\right) \\
\leq \; & 3^{2p-1}\left(K_L(2p,T,\phi)|t-s|^{2p} + K_M(p,T,\phi)|t-s|^p + \|\phi\|_\infty^{2p}|t-s|^{2p}\right) \\
\leq \; & 3^{2p-1}\left(K_L(2p,T,\phi)T^p + K_M(p,T,\phi) + \|\phi\|_\infty^{2p}T^p\right)|t-s|^p \\
\equiv \; & K_Y(p,T,\phi)|t-s|^p,
\end{aligned}$$

for some $K_Y(p,T,\phi)$.

Therefore we have by Theorem 2.3.6, $Y^{\varepsilon_n}(\phi)$ is C-relatively compact in $C(\mathbb{R})$. Suppose $Y^{\varepsilon_n}(\phi)$ converges to some continuous weak limit point $Y(\phi)$, by possibly passing through a subsequence. If d_T represents the metric that induces the Skorohod J_1 topology on $D([0,T])$, then

$$d_T(X^{\varepsilon_n}(\phi), Y^{\varepsilon_n}(\phi)) \leq \sup_{t\leq T}|X_t^{\varepsilon_n}(\phi) - Y_t^{\varepsilon_n}(\phi)| = \sup_{t\leq T}\hat{A}^{\varepsilon_n}(\phi)$$

By Lemma 2.3.10 we have $\sup_{t\leq T}\hat{A}^{\varepsilon_n}(\phi)$ converges to 0 in L^2 and hence weakly. Therefore Slutsky's theorem tells us $X^{\varepsilon_n}(\phi) \to Y(\phi)$ weakly on $[0,T]$. Thus X^{ε_n} is C-relatively compact in $D(\mathbb{R})$.

(b) Theorem 2.3.6 and Lemma 2.3.9(c) tell us that $I^{\varepsilon_n}(\phi)$ is C-relatively compact in $C(\mathbb{R})$. It follows that A^{ε_n} is C-relatively compact in $D(\mathbb{R})$, by identical reasoning as the last paragraph in the proof of (a) by replacing Y^{ε_n} with I^{ε_n} and X^{ε} with A^{ε_n}.

□

2.3.2 Proof of theorem 2.3.4: Compact containment condition

Now that we have shown part (i) of Jakubowski's theorem, it remains to show that $\{X^\varepsilon\}$ and $\{A^{\varepsilon_n}\}$ satisfy the compact containment condition. Before we begin, it will be useful to find an estimate on $\mathbb{E}\left(\bar{X}^{\varepsilon}([-R,R]^c)\right)$ for R large. For the remainder of this section, we fix ϕ_R to be a a smooth approximation to $1([-R,R]^c)$ such that

$$1(R+1<|x|) \leq \phi_R(x) \leq 1(R<|x|).$$

Lemma 2.3.12. *For all $\eta, T>0$ there exists a compact $K_{\eta,T}$ large enough such that for all $t \leq T$*

$$\mathbb{E}\left(\bar{X}^{\varepsilon}_t(K^c_{\eta,T})\right) < \eta,$$

uniformly in ε.

Proof. Since $\bar{X}^{\varepsilon}$ is a solution to (2.10), we have a Green's function representation of (2.10) (which can be deduced from (Perkins, 2002, §II.5)),

$$\bar{X}^{\varepsilon}_t(\phi_R) = \int_0^t \int P_{t-s}\phi_R \bar{A}^{\varepsilon}(ds,dx) + \int_0^t \int P_{t-s}\phi_R(x)\bar{M}(ds,dx), \tag{2.36}$$

Where P_t is the Brownian semigroup. So we have

$$\begin{aligned}
\mathbb{E}\left(\bar{X}^{\varepsilon}_t([-R-1,R+1]^c)\right) &\leq \mathbb{E}\left(\bar{X}^{\varepsilon}_t(\phi_R)\right) \\
&= \mathbb{E}\left(\int_0^t \int P_{t-s}\phi_R \bar{A}^{\varepsilon}(ds,dx)\right) \\
&= \mathbb{E}\left(\int_0^t \int \langle P_{t-s}\phi_R, \phi^x_{\varepsilon}\rangle \Xi^{\varepsilon}(ds,dx)\right) \\
&= \int_0^t \int \langle P_{t-s}\phi_R, \phi^x_{\varepsilon}\rangle \Lambda^{\varepsilon}(ds,dx) \\
&= \int_0^t \int \langle P_{t-s}\phi_R, \phi^x_{\varepsilon}\rangle \frac{1}{\varepsilon}\psi(x)dxds. \qquad (2.37)
\end{aligned}$$

We now need to control $P_{t-s}\phi_R$. If $|x| \leq \frac{R}{2}$,

$$\begin{aligned}
P_t\phi_R(x) &= \int \frac{1}{\sqrt{2\pi t}} e^{-\frac{|x-y|^2}{2t}} \phi_R(y)dy \\
&\leq \int_{|y|\geq R} \frac{1}{\sqrt{2\pi t}} e^{-\frac{|x-y|^2}{2t}} dy \\
&= \int_{|\sqrt{t}z+x|\geq R} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz \\
&\leq \int_{\sqrt{t}|z|+|x|\geq R} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz \\
&= \int_{|z|\geq \frac{R-|x|}{\sqrt{t}}} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz \\
&\leq \int_{|z|\geq \frac{R}{2\sqrt{T}}} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz.
\end{aligned}$$

Where we made the substitution $z = \frac{y-x}{\sqrt{t}}$. By picking R large enough then we have for $|x| \leq \frac{R}{2}$,

$$P_t\phi_R(x) \leq \frac{\eta}{2T}. \tag{2.38}$$

Also since ψ is integrable, we can pick R large enough such that

$$\int_{|x|\geq \frac{R}{2}-1} \psi(x)dx \leq \frac{\eta}{2T}. \tag{2.39}$$

We now return to (2.37),

$$\begin{aligned}
& \int_0^t \int \langle P_{t-s}\phi_R, \phi_\varepsilon^x \rangle \frac{1}{\varepsilon}\psi(x)dxds \\
= \ & \int_0^t \int_{|x|\geq \frac{R}{2}-1} \langle P_{t-s}\phi_R, \phi_\varepsilon^x \rangle \frac{1}{\varepsilon}\psi(x)dxds + \int_0^t \int_{|x|< \frac{R}{2}-1} \langle P_{t-s}\phi_R, \phi_\varepsilon^x \rangle \frac{1}{\varepsilon}\psi(x)dxds \\
\equiv \ & I_1 + I_2.
\end{aligned} \tag{2.40}$$

We will handle both I_1 and I_2 separately.

$$\begin{aligned}
I_1 &= \int_0^t \int_{|x|\geq \frac{R}{2}-1} \langle P_{t-s}\phi_R, \phi_\varepsilon^x \rangle \frac{1}{\varepsilon}\psi(x)dxds \\
&\leq \int_0^t \int_{|x|\geq \frac{R}{2}-1} \|P_{t-s}\phi_R\|_\infty \|\phi_\varepsilon^x\|_1 \frac{1}{\varepsilon}\psi(x)dxds \\
&\leq \int_0^t \int_{|x|\geq \frac{R}{2}-1} \|\phi_R\|_\infty \varepsilon \frac{1}{\varepsilon}\psi(x)dxdt \\
&= \int_0^t \int_{|x|\geq \frac{R}{2}-1} \psi(x)dxds \\
&\leq t\frac{\eta}{2T} \\
&\leq \frac{\eta}{2}. \qquad (2.41)
\end{aligned}$$

The first inequality was by Hölder, and the second was because P is a contractive semigroup. The third inequality was by (2.39). Before we move onto I_2, we should note that for all bounded function f on $\mathbb{R}$,

$$\begin{aligned}
\frac{1}{\varepsilon}\langle f, \phi_\varepsilon^x \rangle &= \int_{x-\sqrt{\varepsilon}}^{x+\sqrt{\varepsilon}} f(y)\frac{\phi_\varepsilon^x(y)}{\varepsilon}dy \\
&\leq \left(\sup_{[x-\sqrt{\varepsilon}, x+\sqrt{\varepsilon}]} f \right) \int \frac{\phi_\varepsilon^x(y)}{\varepsilon}dy \\
&= \sup_{[x-\sqrt{\varepsilon}, x+\sqrt{\varepsilon}]} f \\
&\leq \sup_{[x-1, x+1]} f. \qquad (2.42)
\end{aligned}$$

The last inequality was because we can assume $\varepsilon \leq 1$. Therefore by combining (2.38), and

(2.42) we get,

$$\begin{aligned}
I_2 &= \int_0^t \int_{|x|<\frac{R}{2}-1} \langle P_{t-s}\phi_R, \phi_\varepsilon^x \rangle \frac{1}{\varepsilon} \psi(x) dx dt \\
&\leq \int_0^t \int_{|x|<\frac{R}{2}-1} \left(\sup_{[x-1,x+1]} P_{t-s}\phi_R \right) \psi(x) dx ds \\
&\leq \int_0^t \int_{|x|<\frac{R}{2}-1} \frac{\eta}{2T} \psi(x) dx ds \\
&\leq t \frac{\eta}{2T} \\
&\leq \frac{\eta}{2}.
\end{aligned} \tag{2.43}$$

Finally, combining (2.37), (2.40), (2.41), and (2.43), we get the existence of a $K_{\eta,T} \equiv [R-1, R+1]$ such that, for all $t \leq T$,

$$\mathbb{E}\left(\bar{X}_t^\varepsilon (K_{\eta,T}^c) \right) \leq \eta.$$

□

Before proceeding to the compact containment condition for $\{X^\varepsilon\}$, we will need the following estimates on the mass of L^ε, M^ε, and A^ε.

Lemma 2.3.13. *Suppose $\eta, T, R > 0$. If*

$$\sup_{t \leq T} \mathbb{E}\left(\bar{X}_t^\varepsilon ([-R, R]^c) \right) < \eta,$$

then we have the following:

(a) $\mathbb{E}\left(\sup_{t \leq T} L_t^\varepsilon(\phi_R) \right) \leq \left\| \frac{\Delta \phi_R}{2} \right\|_\infty T\eta,$

(b) $\mathbb{E}\left(\sup_{t \leq T} M_t^\varepsilon(\phi_R) \right) \leq 2\sqrt{T\eta},$

(c) $\mathbb{E}\left(\sup_{t \leq T} A_t^\varepsilon(\phi_R) \right) \leq T \int_{|x|>R-1} \psi(x) dx.$

Proof. (a) We proceed by directly using the definition of L^ε.

$$\begin{aligned}
\mathbb{E}\left(\sup_{t\le T} L_t^\varepsilon(\phi_R)\right) &= \mathbb{E}\left(\sup_{t\le T}\int_0^t X_u^\varepsilon\left(\frac{\Delta\phi_R}{2}\right)du\right)\\
&\le \left\|\frac{\Delta\phi_R}{2}\right\|_\infty \mathbb{E}\left(\int_0^T \bar{X}_u^\varepsilon\left([-R,R]^c\right)du\right)\\
&= \left\|\frac{\Delta\phi_R}{2}\right\|_\infty \int_0^T \mathbb{E}\left(\bar{X}_u^\varepsilon\left([-R,R]^c\right)\right)du\\
&\le \left\|\frac{\Delta\phi_R}{2}\right\|_\infty T\eta.
\end{aligned}$$

(b) By Hölder's inequality and Doob's strong L^2 inequality we get,

$$\begin{aligned}
\mathbb{E}\left(\sup_{t\le T} M_t^\varepsilon(\phi_R)\right) &\le \mathbb{E}\left(\sup_{t\le T} M_t^\varepsilon(\phi_R)^2\right)^{\frac{1}{2}}\\
&\le 2\mathbb{E}\left(M_T^\varepsilon(\phi_R)^2\right)^{\frac{1}{2}}\\
&= 2\mathbb{E}\left(\langle M^\varepsilon(\phi_R)\rangle_T\right)^{\frac{1}{2}}\\
&= 2\mathbb{E}\left(\int_0^T X_s^\varepsilon(\phi_R^2)ds\right)^{\frac{1}{2}}\\
&= 2\left(\int_0^T \mathbb{E}\left(X_s^\varepsilon(\phi_R^2)\right)ds\right)^{\frac{1}{2}}\\
&\le 2\left(\int_0^T \mathbb{E}\left(\bar{X}_s^\varepsilon\left([-R,R]^c\right)\right)ds\right)^{\frac{1}{2}}\\
&\le 2\sqrt{T\eta}.
\end{aligned}$$

(c) Finally,

$$\begin{aligned}
\mathbb{E}\left(\sup_{t\le T} A_t^\varepsilon(\phi_R)\right) &\le \mathbb{E}\left(\sup_{t\le T} \bar{A}_t^\varepsilon(\phi_R)\right)\\
&= \mathbb{E}\left(\sup_{t\le T}\int_0^t\int\langle \phi_R, \phi_\varepsilon^x\rangle \Xi^\varepsilon(ds,dx)\right)\\
&\le \mathbb{E}\left(\int_0^T\int\langle \phi_R, \phi_\varepsilon^x\rangle \Xi^\varepsilon(ds,dx)\right)\\
&= \int_0^T\int\langle \phi_R, \phi_\varepsilon^x\rangle \Lambda^\varepsilon(ds,dx)\\
&= \int_0^T\int\langle \phi_R, \phi_\varepsilon^x\rangle \frac{1}{\varepsilon}\psi(x)dxds\\
&= T\int\langle \phi_R, \phi_\varepsilon^x\rangle \frac{1}{\varepsilon}\psi(x)dx.
\end{aligned}$$

Since we are taking limits as $\varepsilon \to 0$, we can assume $\varepsilon < 1$. Therefore by (2.42) we have,

$$\frac{1}{\varepsilon}\langle \phi_R, \phi_\varepsilon^x\rangle \le \sup_{|y|\le |x|+\sqrt{\varepsilon}} \phi_R(y) \le \sup_{|y|\le|x|+1} 1(R<|x|) = 1(R-1<|x|).$$

So we have shown,

$$\mathbb{E}\left(\sup_{t\le T} A_t^\varepsilon(\phi_R)\right) \le T\int_{|x|\ge R-1}\psi(x)dx.$$

□

We now prove that $\{X^\varepsilon\}$ and $\{A^\varepsilon\}$ satisfy the compact containment condition.

Proposition 2.3.14. *$\{X^\varepsilon\}$ and $\{A^\varepsilon\}$ satisfies the compact containment condition. I.e, For all $\eta, T > 0$, there is a compact $K_{\eta,T}$ such that,*

(a) $\sup_\varepsilon \mathbb{P}\left(\sup_{t\le T} X_t^\varepsilon(K_{\eta,T}^c) > \eta\right) < \eta,$

(b) $\sup_\varepsilon \mathbb{P}\left(\sup_{t\le T} A_t^\varepsilon(K_{\eta,T}^c) > \eta\right) < \eta.$

Proof. (a) Let $\eta, T > 0$. By Lemma 2.3.12 we can find R large enough, depending only on η^* and T such that,

$$\sup_{t \le T} \mathbb{E}\left(\bar{X}_t^\varepsilon([-R, R]^c)\right) < \eta^*.$$

Where $\eta^* > 0$ will be decided later. By using $1(R + 1 < |x|) \le \phi_R(x)$ and Lemma 2.3.13, we have

$$\begin{aligned}
& \mathbb{P}\left(\sup_{t \le T} X^\varepsilon([-R-1, R+1]^c) > \eta\right) \\
\le\ & \mathbb{P}\left(\sup_{t \le T} X^\varepsilon(\phi_R) > \eta\right) \\
\le\ & \frac{\mathbb{E}\left(\sup_{t \le T} X^\varepsilon(\phi_R)\right)}{\eta} \\
\le\ & \frac{1}{\eta}\left[\mathbb{E}\left(\sup_{t \le T} L_t^\varepsilon(\phi_R)\right) + \mathbb{E}\left(\sup_{t \le T} M_t^\varepsilon(\phi_R)\right) + \mathbb{E}\left(\sup_{t \le T} A_t^\varepsilon(\phi_R)\right)\right] \\
\le\ & \frac{1}{\eta}\left(\left\|\frac{\Delta \phi_K}{2}\right\|_\infty T\eta^* + 2\sqrt{T\eta^*} + T\int_{|x|>R-1} \psi(x)dx\right). \qquad (2.44)
\end{aligned}$$

Since ψ is integrable, we can pick R large enough so that

$$\int_{|x|>R-1} \psi(x)dx < \eta^*. \tag{2.45}$$

Therefore by letting η^* be small enough, such that (2.44) is less than η. Therefore we have found a compact set $K_{\eta,T} \equiv [-R-1, R+1]$, such that,

$$\mathbb{P}\left(\sup_{t \le T} X^\varepsilon(K_{\eta,T}^c) > \eta\right) < \eta, \tag{2.46}$$

independent of ε.

(b) Let R and $K_{\eta,T}$ be as above. By using $1(|x| > R+1) \le \phi_R(x)$ and Lemma 2.3.13, we have

$$\begin{aligned}
\mathbb{P}\left(\sup_{t\leq T} A^{\varepsilon}(K^c_{\eta,T}) > \eta\right) &\leq \mathbb{P}\left(\sup_{t\leq T} A^{\varepsilon}(\phi_R) > \eta\right) \\
&\leq \frac{\mathbb{E}\left(\sup_{t\leq T} A^{\varepsilon}(\phi_R)\right)}{\eta} \\
&\leq \frac{T}{\eta}\int_{|x|>R-1} \psi(x)dx \qquad (2.47) \\
&< \eta,
\end{aligned}$$

independent of ε. The last inequality was because (2.47) was smaller than (2.44) which is less than η.

□

Thus we have finally proven Theorem 2.3.4.

2.4 SPDE characterization of limit points

We now have by Theorem 2.3.4 the existence of a sequence $\varepsilon_n \to 0$ as $n \to \infty$ and continuous M_F-valued processes X, A such that $(X^{\varepsilon_n}, A^{\varepsilon_n})$ converge weakly to (X, A), possibly by taking subsequences. Let $\mathcal{F}_t$ be the filtration generated by the processes X and A, ie.,

$$\mathcal{F}_t = \sigma\left(X_s, A_s | s \leq t\right).$$

We want to show that for all $\phi \in C_b^\infty$, X admits an SPDE representation of the form,

$$X_t(\phi) = L_t(\phi) + M_t(\phi) + A_t(\phi).$$

Where

$$L_t(\phi) \equiv \int_0^t X_s\left(\frac{\Delta\phi}{2}\right) ds,$$

and $M(\phi)$ is a continuous $(\mathcal{F}_t)$-martingale with quadratic variation $\langle M(\phi)\rangle_t = \int_0^t X_s(\phi^2)ds$.

By rearranging the terms in (2.9), we have for all $\phi \in C_b^\infty$,

$$M_t^{\varepsilon_n}(\phi) = X_t^{\varepsilon_n}(\phi) - L_t^{\varepsilon_n}(\phi) - A_t^{\varepsilon_n}(\phi), \qquad (2.48)$$

is a $(\mathcal{F}_t^{\varepsilon_n})$-martingale with quadratic variation $\langle M^{\varepsilon_n}\rangle_t = \int_0^t X_s^{\varepsilon_n}(\phi^2)ds$. The plan of attack to argue we get our desired $M(\phi)$ by taking limits of M^{ε_n} as $n \to \infty$. Before we proceed, it will be convenient to work with almost sure convergence as opposed to weak.

Theorem 2.4.1 (Skorohod Representation Theorem)**.** *(Billingsley, 2013, p. 70) Let (S, d) be a Polish space, and suppose X^n, X, be S-valued random variables (on possibly different probability spaces) such that X^n converges weakly to X. There exists a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ and random variables $\tilde{X}^n, \tilde{X}$ with the same law as X^n, X and $\tilde{\mathbb{P}}$-a.s,*

$$\tilde{X}^n(\omega) \xrightarrow[n\to\infty]{(S,d)} \tilde{X}(\omega).$$

For the remainder of this paper we will assume that without loss of generality that we are working on a common probability space where $(X^{\varepsilon_n}, A^{\varepsilon_n})$ converges a.s. to (X, A) in $D(M_F) \times D(M_F)$.

Note that for all $\phi \in C_b^\infty$, $X \to X(\phi)$ in a continuous map from $D(M_F) \to D(\mathbb{R})$ and $X \to \int_0^\cdot X_s\left(\frac{\Delta\phi}{2}\right) ds$ is a continuous map from $D(M_F) \to C(\mathbb{R})$. This implies for all $\phi \in C_b^\infty$, a.s.

$$\lim_{n\to\infty} X_t^{\varepsilon_n}(\phi) = X_t(\phi),$$
$$\lim_{n\to\infty} A_t^{\varepsilon_n}(\phi) = A_t(\phi),$$

and,

$$\lim_{n\to\infty} L_t^{\varepsilon_n}(\phi) = \lim_{n\to\infty} \int_0^t X_s^{\varepsilon_n}\left(\frac{\Delta\phi}{2}\right) ds = \int_0^t X_s\left(\frac{\Delta\phi}{2}\right) ds = L_t(\phi).$$

Therefore a.s.,

$$\begin{aligned}\lim_{n\to\infty} M_t^{\varepsilon_n}(\phi) &= \lim_{n\to\infty} X_t^{\varepsilon_n}(\phi) - L_t^{\varepsilon_n}(\phi) - A_t^{\varepsilon_n}(\phi) \\ &= X_t(\phi) - L_t(\phi) - A_t(\phi) \\ &\equiv M_t(\phi) \qquad (2.49)\end{aligned}$$

If $N^{\varepsilon_n}(\phi) \equiv M^{\varepsilon_n}(\phi)^2 - \langle M^{\varepsilon_n}(\phi)\rangle$ is the square $(\mathcal{F}_t^{\varepsilon_n})$-martingale for $M^{\varepsilon_n}(\phi)$. Similar to above, for all $\phi \in C_b^\infty$, $X \to \int_0^t X_s(\phi^2)ds$ is a continuous map from $D(M_F) \to C(\mathbb{R})$, which implies a.s.

$$\begin{aligned}\lim_{n\to\infty} N_t^{\varepsilon_n}(\phi) &= \lim_{n\to\infty} M_t^{\varepsilon_n}(\phi)^2 - \langle M^{\varepsilon_n}(\phi)\rangle_t \\ &= \lim_{n\to\infty} M_t^{\varepsilon_n}(\phi)^2 - \int_0^t X_s^{\varepsilon_n}(\phi^2)ds \\ &= M_t(\phi)^2 - \int_0^t X_s(\phi^2)ds \\ &\equiv N_t(\phi). \end{aligned} \tag{2.50}$$

Since X, L and A are continuous, we have that $M(\phi)$ and $N(\phi)$ are also continuous. It remains to show that $M(\phi)$ and $N(\phi)$ are $(\mathcal{F}_t)$-martingales. Note that for all $T > 0$, Burkholder-Davis-Gundy inequality and Jensen's inequality for x^2 gives us,

$$\begin{aligned}\mathbb{E}\left(\sup_{t\le T} M_t^{\varepsilon_n}(\phi)^4\right) &\le C\mathbb{E}\left(\langle M^{\varepsilon_n}(\phi)\rangle_T^2\right) \\ &= C\mathbb{E}\left(\int_0^T X_s^{\varepsilon_n}(\phi^2)ds\right)^2 \\ &= CT\mathbb{E}\int_0^T X_s^{\varepsilon_n}(\phi^2)^2 ds \\ &\le CT\|\phi\|_\infty^4 \mathbb{E}\int_0^T \bar{X}_s^{\varepsilon_n}(1)^2 ds \\ &= CT\|\phi\|_\infty^4 \int_0^T \mathbb{E}\left(\bar{X}_s^{\varepsilon_n}(1)^2\right) ds \end{aligned} \tag{2.51}$$

We can estimate the integral in (2.51) by applying Lemma 2.3.8,

$$\begin{aligned}\mathbb{E}\left(\sup_{t\le T} M_t^{\varepsilon_n}(\phi)^4\right) &\le CT\|\phi\|_\infty^4 \int_0^T C(2,T)s^2 + \delta_{2,T}(\varepsilon_n)ds \\ &= CT\|\phi\|_\infty^4 \left(\frac{C(2,T)T^3}{3} + T\delta_{2,T}(\varepsilon_n)\right) \\ &\le C\|\phi\|_\infty^4 \left(\frac{C(2,T)T^4}{3} + TD_{2,T}\right) \\ &< \infty \end{aligned}$$

The last line used the fact that $\delta_{2,T}(\varepsilon_n) \to 0$ as $n \to \infty$, and is thus uniformly bounded in n by some $D_{2,T}$.

Hence we have shown that $M_t^{\varepsilon_n}(\phi)$ and $N_t^{\varepsilon_n}(\phi)$ are L^4 and L^2 bounded respectively uniformly in n and $0 \le t \le T$. Therefore $\{\sup_{t \le T} M_t^{\varepsilon_n}(\phi)\}$ and $\{\sup_{t\le T} N_t^{\varepsilon_n}(\phi)\}$ are uniformly integrable, which along with (2.49) and (2.50) imply $M^{\varepsilon_n}(\phi) \to M(\phi)$ and $N^{\varepsilon_n}(\phi) \to N(\phi)$ in L^1.

L^1 convergence and the fact that $M^{\varepsilon_n}(\phi)$ and $N^{\varepsilon_n}(\phi)$ are $(\mathcal{F}_t^{\varepsilon_n})$-martingales, imply for all $\Psi : M_F^{2m} \to \mathbb{R}$ be bounded and continuous and $s_1, \dots s_m, s \le t$,

$$\begin{aligned}
0 &= \lim_{n\to\infty} \mathbb{E}\left((M_t^{\varepsilon_n}(\phi) - M_s^{\varepsilon_n}(\phi))\Psi(X_{s_1}^{\varepsilon_n}, \dots X_{s_m}^{\varepsilon_n}, A_{s_1}^{\varepsilon_n}, \dots, A_{s_m}^{\varepsilon_n})\right) \\
&= \mathbb{E}\left(\lim_{n\to\infty} (M_t^{\varepsilon_n}(\phi) - M_s^{\varepsilon_n}(\phi))\Psi(X_{s_1}^{\varepsilon_n}, \dots X_{s_m}^{\varepsilon_n}, A_{s_1}^{\varepsilon_n}, \dots, A_{s_m}^{\varepsilon_n})\right) \\
&= \mathbb{E}\left((M_t(\phi) - M_s(\phi))\Psi(X_{s_1}, \dots X_{s_m}, A_{s_1}, \dots, A_{s_m})\right). \qquad (2.52)
\end{aligned}$$

Similarly,

$$\begin{aligned}
0 &= \lim_{n\to\infty} \mathbb{E}\left((N_t^{\varepsilon_n}(\phi) - N_s^{\varepsilon_n}(\phi))\Psi(X_{s_1}^{\varepsilon_n}, \dots X_{s_m}^{\varepsilon_n}, A_{s_1}^{\varepsilon_n}, \dots, A_{s_m}^{\varepsilon_n})\right) \\
&= \mathbb{E}\left(\lim_{n\to\infty} (N_t^{\varepsilon_n}(\phi) - N_s^{\varepsilon_n}(\phi))\Psi(X_{s_1}^{\varepsilon_n}, \dots X_{s_m}^{\varepsilon_n}, A_{s_1}^{\varepsilon_n}, \dots, A_{s_m}^{\varepsilon_n})\right) \\
&= \mathbb{E}\left((N_t(\phi) - N_s(\phi))\Psi(X_{s_1}, \dots X_{s_m}, A_{s_1}, \dots, A_{s_m})\right). \qquad (2.53)
\end{aligned}$$

Therefore we can conclude by (2.52), and (2.53) that $M(\phi)$ and $N(\phi)$ are $(\mathcal{F}_t)$-martingales. Since $N_t(\phi) = M_t(\phi)^2 - \int_0^t X_s(\phi^2)ds$ is a $(\mathcal{F}_t)$-martingale, we have by the uniqueness of quadratic variation, $\langle M(\phi)\rangle_t = \int_0^t X_s(\phi^2)ds$. We have proven the following.

Theorem 2.4.2. *If (X, A) is a weak limit point of $(X^{\varepsilon_n}, A^{\varepsilon_n})$ as $\varepsilon_n \to 0$, then for all $\phi \in C_b^\infty(\mathbb{R})$, X satisfies the SPDE,*

$$X_t(\phi) = L_t(\phi) + M_t(\phi) + A_t(\phi). \qquad (2.54)$$

Where $M(\phi)$ is a continuous $(\mathcal{F}_t)$-martingale with quadratic variation,

$$\langle M(\phi)\rangle_t = \int_0^t X_s(\phi^2)ds.$$

2.5 Density of X

We now have the existence of a sequence $\varepsilon_n \to 0$ as $n \to \infty$, and continuous M_F-valued processes X, that satisfies the SPDE (2.54), such that X^{ε_n} converges weakly to X.

Recall we have a process u^{ε_n} with sample paths in $D(C_0)$ defined by (2.5), satisfying $X_t^{\varepsilon_n}(dx) = u_t^{\varepsilon_n}(x)dx$. We will now show that there is a process u such with sample paths in $C(C_0)$ and satisfies $X_t(dx) = u_t(x)dx$. We will show that u can be determined as weak limit point of u^{ε_n}, by the following theorem.

Theorem 2.5.1. *If $\varepsilon_n \to 0$ is defined as above, then $\{u^{\varepsilon_n}\}$ is C-relatively compact in $D(C_0)$.*

An immediate consequence of Theorem 2.5.1 is we get the existence of our process continuous C_0-valued process u.

Corollary 2.5.2. *If u^{ε_n} converges weakly to u, then a.s. $X_t(dx) = u(t,x)dx$. In particular this implies that the limit points are a.s. unique.*

Proof. By Theorem 2.5.1 and Skorohod representation theorem, we can assume without loss of generality that u^{ε_n} converges to u a.s as C_0-valued processes, possibly by taking a subsequence. This implies on all compact sets, u^{ε_n} to u uniformly. Then for all $\phi \in C_c^\infty$,

$$\begin{aligned}\int \phi(x)u(t,x)dx &= \int \lim_{n\to\infty} \phi(x)u^{\varepsilon_n}(t,x)dx \\ &= \lim_{n\to\infty} \int \phi(x)u^{\varepsilon_n}(t,x)dx \\ &= \lim_{n\to\infty} X_t^{\varepsilon_n}(\phi) \\ &= X_t(\phi).\end{aligned}$$

The exchange of limits was justified because u^{ε_n} converges to u uniformly on the support of ϕ. The last line used the fact that $X_t^{\varepsilon_n}$ convergences weakly to X. Thus X_t is absolutely continuous with respect to Lebesgue measure and $X_t(dx) = u(t,x)dx$. □

We now have the existence of sequence $\varepsilon_n \to 0$ as $n \to \infty$, and continuous M_F-valued process X, A, and C_0-valued process u such that

$$(X^{\varepsilon_n}, A^{\varepsilon_n}, u^{\varepsilon_n}) \to (X, A, u),$$

weakly, possibly by taking subsequence. By Skorohod's representation theorem, we can assume without loss of generality that all the random variables are defined on a common probability space and the convergence is a.s. Finally by taking subsequences, we can further assume that

$$\sum_n \varepsilon_n < \infty. \tag{2.55}$$

It remains to justify Theorem 2.5.1 is true. We will do this in the following section.

2.5.1 Proof of Theorem 2.5.1: C-relative compactness of $\{u^\varepsilon\}$

The proof of Theorem 2.5.1 is quite technical so we will just give an outline. Recall that,

$$\phi_\varepsilon^x(y) = \sqrt{\varepsilon}\phi\left(\frac{y-x}{\sqrt{\varepsilon}}\right).$$

Where $\phi : \mathbb{R} \to \mathbb{R}_+$ is a Lipschitz functions supported on $[-1,1]$ satisfying $\|\phi\|_1 = 1$, and $\|\phi\|_\infty \leq 1$. For now instead of working on C_0 we will work on $C(\mathbb{R})$ the set of continuous functions from $\mathbb{R}$ to $\mathbb{R}$, equipped with the compact-open topology.

Our main tool to show $\{u^\varepsilon\}$ is C-relatively compact in $D(C_0)$ will be the following modification of Theorem 2.3.6.

Theorem 2.5.3. *Suppose $\{Y^n\}$ is a family of stochastic processes with sample paths in $C(C(\mathbb{R}))$ and $p > 1$ and $a, b > 2$ such that for all $T > 0$, and there is a $C(T) > 0$ satisfying*

$$\mathbb{E}\left(|Y_t^\varepsilon(x) - Y_{t'}^\varepsilon(x')|^p\right) \leq C(T)\left(|t-t'|^a + |x-x'|^b\right),$$

for $0 \leq t, t' \leq T$ and $|x|, |x'| \leq T$. Then $\{Y^n\}$ is C-relatively compact in $C(C(\mathbb{R}))$.

Recall if u^i is density of SBM X^i with initial law $\phi_\varepsilon^{x_i}(z)dz$, then Theorem III.4.2 in (Perkins, 2002, §III.4) showed that the Green's function representation for u^i is,

$$u_t^i(x) = \int_0^t \int P_{t-s}(y-x) M^i(ds, dy) + P_t \phi_\varepsilon^{x_i}(x). \tag{2.56}$$

By substituting (2.56) into (2.5), we get the following Green's function representation of u^ε.

$$u_t^\varepsilon(x) = \int_0^t \int P_{t-s}(y-x) M^\varepsilon(ds, dy) + \int_0^t \int P_{t-s}\phi_\varepsilon^y(x) 1(u_{s-}^\varepsilon(y) = 0) \Xi^\varepsilon(ds, dy)$$
$$\equiv N_t^\varepsilon(x) + H_t^\varepsilon(x) \tag{2.57}$$

We further decompose $H_t^\varepsilon(x)$ using the compensated PPP to get,

$$H_t^\varepsilon(x) = \int_0^t \int P_{t-s}\phi_\varepsilon^y(x) 1(u_{s-}^\varepsilon(y) = 0) \Lambda^\varepsilon(ds, dy) + \int_0^t \int P_{t-s}\phi_\varepsilon^y(x) 1(u_{s-}^\varepsilon(y) = 0) \hat{\Xi}^\varepsilon(ds, dy)$$
$$\equiv h_t^\varepsilon(x) + \hat{H}_t^\varepsilon(x). \tag{2.58}$$

So we have,

$$u_t^\varepsilon(x) = N_t^\varepsilon(x) + h_t^\varepsilon(x) + \hat{H}_t^\varepsilon(x), \tag{2.59}$$

where $\hat{H}^\varepsilon(x)$ is a jump martingale. We will deal with each term individually. The analysis in Theorem III.4.2 in (Perkins, 2002, § III.4), shows that $\{N^\varepsilon\}$ is C-relatively compact in $C(C(\mathbb{R}))$. To deal with h^ε we note the following estimate.

Lemma 2.5.4. *Let $T > 0$. Then there is a $C(T)$ such that for all $0 \le t' < t \le T$ and $|x|, |x'| \le T$,*

$$|h_t^\varepsilon(x) - h_{t'}^\varepsilon(x')| \le C(T) \left(\sqrt{t - t'} + |x - x'| \right).$$

Proof. This is a lengthy computation. □

Lemma 2.5.4 in combination with Theorem 2.5.3 gives us h^ε is also C-relatively compact in $C(C(\mathbb{R}))$. It remains to deal with $\hat{H}^\varepsilon$. Burkholder's inequality for jump processes eventually yields the following two estimates.

Lemma 2.5.5. *Let $p > 1$ and $T > 0$. There is a $C(T) > 0$ such that for all $0 \le t \le T$ and $|x|, |x'| \le T$,*

$$\mathbb{E}\left(|\hat{H}_t^\varepsilon(x) - \hat{H}_t^\varepsilon(x')|^p \right) \le C(T) \varepsilon^{\frac{p}{4}} |x - x'|^{\frac{p}{2}}.$$

Lemma 2.5.6. *Let $p > 1$ and $T > 0$. There is a $C(T) > 0$ such that for all $0 \le t' < t \le T$ and $|x| \le T$,*

$$\mathbb{E}\left(|\hat{H}_t^\varepsilon(x) - \hat{H}_{t'}^\varepsilon(x)|^p \right) \le C(T) \left(|t - t'|^{\frac{p}{4}} + \varepsilon^{\frac{p}{2}} \right).$$

The discontinuity in t of H^ε prevents us from directly applying Theorem 2.5.3. To fix this issue we will approximate H^ε with the continuous process $\tilde{H}^\varepsilon$ defined as follows. For all $t \in \{i\varepsilon^q | i \in \mathbb{N}\} \equiv G_{\varepsilon,q}$ for $q > 1$,

$$\tilde{H}^\varepsilon_t(x) = H^\varepsilon_t(x). \tag{2.60}$$

For $t \notin G_{\varepsilon,q}$ we linearly interpolate in t. Now Lemma 2.5.4, 2.5.5, and 2.5.6 imply that for all $t, t' \in G_{\varepsilon,q}$, such that $0 \leq t' < t \leq T$ we have a $C(T)$,

$$\begin{aligned}\mathbb{E}\left(|\tilde{H}^\varepsilon_t(x) - \tilde{H}^\varepsilon_{t'}(x')|^p\right) &\leq C(T)\left(|x - x'|^{\frac{p}{2}} + |t - t'|^{\frac{p}{2}} + \varepsilon^{\frac{p}{2}}\right) \\ &= C(T)\left(|x - x'|^{\frac{p}{2}} + |t - t'|^{\frac{p}{2}} + (\varepsilon^q)^{\frac{p}{2q}}\right) \\ &\leq C(T)\left(|x - x'|^{\frac{p}{2}} + |t - t'|^{\frac{p}{2}} + |t - t'|^{\frac{p}{2q}}\right) \\ &\leq C'(T)\left(|x - x'|^{\frac{p}{2}} + |t - t'|^{\frac{p}{2q}}\right).\end{aligned} \tag{2.61}$$

The last line used the fact that $q > 1$. By linear interpolation we have (2.61) holds for all $0 \leq t' \leq t \leq T$. Therefore Theorem 2.5.3 applied and we have $\tilde{H}^\varepsilon$ is C-relatively compact in $C(C(\mathbb{R}))$. To show that H^ε is C-relatively compact, by Slutsky's theorem it suffices to show that $H^\varepsilon - \tilde{H}^\varepsilon$ converges to 0 weakly. Let $x \in \mathbb{R}$, $0 \leq t \leq T$, and suppose that ψ is compactly supported.

$$\begin{aligned}&|H^\varepsilon_t(x) - \tilde{H}^\varepsilon_t(x)| \\ \leq\ & C \sup_{i\varepsilon^q \leq T, t \in [i\varepsilon^q, (i+1)\varepsilon^q]} |H^\varepsilon_t(x) - H^\varepsilon_{i\varepsilon^q}(x)| \\ \leq\ & C \sup_{i\varepsilon^q \leq T, t \in [i\varepsilon^q, (i+1)\varepsilon^q]} \int_0^{i\varepsilon^q} \int |P_{t-s}\phi^y_\varepsilon(x) - P_{i\varepsilon^q - s}\phi^y_\varepsilon(x)| \Xi^\varepsilon(ds, dx) \\ & + C \sup_{i\varepsilon^q \leq T, t \in [i\varepsilon^q, (i+1)\varepsilon^q]} \int_{i\varepsilon^q}^t \int P_{t-s}\phi^y_\varepsilon(x) \Xi^\varepsilon(ds, dx)\end{aligned} \tag{2.62}$$

Now one can show that the difference in the first integrand of (2.62) is bounded by $\varepsilon^{\frac{q}{2}}$ and the using the fact that P is a contractive and $\|\phi^y_\varepsilon\|_\infty \leq \sqrt{\varepsilon}$, we get the second integrand is bounded by $\sqrt{\varepsilon}$. So (2.62) becomes,

$$\sup_{x\in\mathbb{R},t\le T} |H_t^\varepsilon(x) - \tilde{H}_t^\varepsilon(x)| \le C\left(\varepsilon^{\frac{q}{2}}\Xi^\varepsilon([0,T]\times\mathbb{R}) + \sqrt{\varepsilon}\sup_{i\varepsilon^q\le T}\Xi^\varepsilon([i\varepsilon^q,(i+1)\varepsilon^q]\times\mathbb{R})\right) \tag{2.63}$$

We can show that (2.63) goes to 0 is probability uniformly on $[0,T]\times\mathbb{R}$. Hence we have proven that $\{u^\varepsilon\}$ is C-relatively compact in $D(C(\mathbb{R}))$. By using the compactness of ψ along with the modulus of continuity of SBM derived in (Perkins, 2002, §III.1), we can bootstrap to get that $\{u^\varepsilon\}$ is C-relatively compact in $D(C_0)$.

2.6 Domination of Immigration

Recall that the SBM populations X^i born at time t_i with initial law $\phi_\varepsilon^{x_i}(x)dx$ only contribute to X^ε if the previously contributing populations do not occupy any space at x_i. Therefore A^ε only contribute to the population when,

$$1(u^\varepsilon_{t_i-}(x_i) = 0).$$

Our goal will be to show the same holds in the case of A. In this section we will justify A only contributes to the population X whenever X only at the locations X occupies no mass. In other words we will show that the M_F-valued process A satisfies,

$$A_t(dx) \ll 1(u_t(x) = 0)dx.$$

We do this in the following theorem.

Theorem 2.6.1. *For all non-negative $\phi \in C_b^\infty$,*

$$A_t(\phi) \le \int_0^t\int \phi(x)\psi(x)1(u_s(x)=0)dxds. \tag{2.64}$$

Before we prove Theorem 2.6.1, we will need the following lemma.

Lemma 2.6.2. *Let f be a continuous function, then*

$$\lim_{\varepsilon\to 0}\frac{\langle f,\phi_\epsilon^x\rangle}{\varepsilon} = f(x).$$

Proof. Using the fact that $\varepsilon = \|\phi_\varepsilon^x\|_1$, we have

$$\frac{1}{\varepsilon}\int \phi_\varepsilon^x(y)dy = 1.$$

Let $S_\varepsilon^x \equiv \text{supp}(\phi_\varepsilon^x) = [x-\sqrt{\varepsilon}, x+\sqrt{\varepsilon}]$. We we now compute the limit.

$$\begin{aligned}
\left|\frac{\langle f, \phi_\epsilon^x\rangle}{\varepsilon} - f(x)\right| &= \left|\int f(y)\frac{\phi_\varepsilon^x(y)}{\varepsilon}dy - f(x)\right| \\
&= \left|\int (f(y)-f(x))\frac{\phi_\varepsilon^x(y)}{\varepsilon}dy\right| \\
&\leq \int |f(y)-f(x)|\frac{\phi_\varepsilon^x(y)}{\varepsilon}dy \\
&\leq \left(\sup_{S_\varepsilon^x} f - \inf_{S_\varepsilon^x} f\right)\int \frac{\phi_\varepsilon^x(y)}{\varepsilon}dy \\
&= \left(\sup_{S_\varepsilon^x} f - \inf_{S_\varepsilon^x} f\right).
\end{aligned}$$

Since f is continuous we have that as $\varepsilon \to 0$,

$$\lim_{\varepsilon\to 0}\sup_{S_\varepsilon^x} f = \lim_{\varepsilon\to 0}\inf_{S_\varepsilon^x} f,$$

and thus

$$\lim_{\varepsilon\to 0}\frac{\langle f, \phi_\epsilon^x\rangle}{\varepsilon} = f(x).$$

□

With the lemma in hand, we are now ready to prove Theorem 2.6.1.

Proof of Theorem 2.6.1. We know from (2.29) that

$$A^{\varepsilon_n}(\phi) = I^{\varepsilon_n}(\phi) + \hat{A}^{\varepsilon_n}(\phi). \tag{2.65}$$

So we have for all $T > 0$, by Doob's strong L^2 inequality,

$$\begin{aligned}
& \sum_n \mathbb{E}\left(\sup_{t\leq T} \hat{A}_t^{\varepsilon_n}(\phi)^2)\right) \\
\leq\ & 4\sum_n \mathbb{E}\left(\hat{A}_T^{\varepsilon_n}(\phi)^2\right) \\
=\ & \sum_n \mathbb{E}\left(\int_0^T \int \langle \phi, \phi_{\varepsilon_n}^x \rangle 1(u_s^{\varepsilon_n}(x)=0)\hat{\Xi}^{\varepsilon_n}(ds,dx)\right)^2 \\
=\ & \sum_n \int_0^T \int \langle \phi, \phi_{\varepsilon_n}^x \rangle^2 1(u_s^{\varepsilon_n}(x)=0)\Lambda^{\varepsilon_n}(ds,dx) \\
\leq\ & \sum_n \int_0^T \int \|\phi\|_\infty^2 \varepsilon_n^2 \frac{1}{\varepsilon_n}\psi(x)dxds \\
=\ & \|\phi\|_\infty^2 T \sum_n \varepsilon_n \\
<\ & \infty.
\end{aligned} \tag{2.66}$$

The last line was because of (2.55). By applying Fubini's theorem, we can exchange the sum and expectation in (2.66) to get,

$$\mathbb{E}\left(\sum_n \sup_{t\leq T} \hat{A}_t^{\varepsilon_n}(\phi)^2\right) < \infty.$$

This implies that $\sum_n \sup_{t\leq T} \hat{A}_t^{\varepsilon_n}(\phi)^2$ is finite a.s. and thus, a.s.

$$\lim_{n\to\infty} \sup_{t\leq T} \hat{A}_t^{\varepsilon_n}(\phi) = 0. \tag{2.67}$$

By (2.65) and (2.67) a.s.,

$$A_t(\phi) = \lim_{n\to\infty} A^{\varepsilon_n}(\phi) = \lim_{n\to\infty} I^{\varepsilon_n}(\phi).$$

We will now simplify the right hand side, and use Fatou's lemma.

$$
\begin{aligned}
A(\phi) &= \lim_{n\to\infty} I^{\varepsilon_n}(\phi) \\
&= \lim_{n\to\infty} \int_0^t \int \langle \phi, \phi^x_{\varepsilon_n}\rangle 1(u^{\varepsilon_n}_{s-}(x) = 0)\Lambda^{\varepsilon_n}(ds, dx) \\
&= \lim_{n\to\infty} \int_0^t \int \langle \phi, \phi^x_{\varepsilon_n}\rangle \frac{1}{\varepsilon_n}\psi(x) 1(u^{\varepsilon_n}_{s-}(x) = 0) dxds \\
&\le \int_0^t \int \limsup_{n\to\infty} \langle \phi, \phi^x_{\varepsilon_n}\rangle \frac{1}{\varepsilon_n}\psi(x) 1(u^{\varepsilon_n}_{s-}(x) = 0) dxds \qquad (2.68)\\
&\le \int_0^t \int \lim_{n\to\infty} \left(\langle \phi, \phi^x_{\varepsilon_n}\rangle \frac{1}{\varepsilon_n}\right) \psi(x) \limsup_{n\to\infty} 1(u^{\varepsilon_n}_{s-}(x) = 0) dxds \\
&= \int_0^t \int \phi(x)\psi(x) \limsup_{n\to\infty} 1(u^{\varepsilon_n}_{s-}(x) = 0) dxds. \qquad (2.69)
\end{aligned}
$$

We have (2.68) is true by the reverse Fatou lemma applied to the measure $1(s \le t)\psi(x)dxds$, and the last line was by Lemma 2.6.2. It remains to deal with the lim sup term. Since u^{ε_n} converges to u a.s. as C_0-valued processes, we have u^{ε_n} convergences to u uniformly on compact sets and hence point-wise. This coupled with the fact that u is continuous, we have a.s. $u^{\varepsilon_n}_{s-}(x) \to u_s(x)$ as $n \to \infty$. Since the indicator of a closed set is upper semi-continuous, and u is continuous,

$$\limsup_{n\to\infty} 1(u^{\varepsilon_n}_{s-}(x) = 0) \le 1(u_s(x) = 0).$$

Therefore (2.69) becomes,

$$A(\phi) \le \int_0^t \int \phi(x)\psi(x) 1(u_s(x) = 0) dxds.$$

□

So we have that A is a continuous increasing process which only increases on the zero set of X. There we have shown that X is SBM with immigration A only active where the population X does not occupy space.

2.7 Non-triviality of X

Up until this point we have shown the existence of a continuous M_F-valued process X that satisfies the SPDE, for $\phi \in C_b^\infty$,

$$X_t(\phi) = L_t(\phi) + M_t(\phi) + A_t(\phi). \tag{2.70}$$

Where $M(\phi)$ is a continuous $(\mathcal{F}_t)$-martingale with quadratic variation $\langle M(\phi)\rangle_t = \int_0^t X_s(\phi^2)ds$, and

$$A(ds, dx) \leq \psi(x)1(X(s,x) = 0)dsdx.$$

Even though we have shown the existence of X, as stated, $X = 0$ is a potential solution to (2.70). The remainder of this section will be devoted to ruling out this possibility and showing that X is indeed non-trivial.

By definition, X^ε is the sum of independent SBM clusters with initial mass ε, it will be useful to know how long these clusters live for.

Proposition 2.7.1 (Distribution of lifetime)**.** *Let τ denote the lifetime of a SBM X with initial mass ε. Then τ has a density given by,*

$$\mathbb{P}(\tau \leq t) = \int_0^t \frac{2\varepsilon}{s^2} e^{-\frac{2\varepsilon}{s}} ds.$$

Proof. A proof of this result is given in (Perkins, 2002, p. 171) by analysing the Laplace functional of Super Brownian motion. □

If $(t_i, x_i) \sim \Xi^\varepsilon$ is the time and position of a SBM cluster X^i with initial law $\phi_\varepsilon^{x_i}(z)dz$, then we have τ_i, the lifetime of cluster X^i is independent of (t_i, x_i). Therefore we can extend Ξ^ε to a new PPP $\tilde{\Xi}^\varepsilon$ on $[0,\infty) \times \mathbb{R} \times (0,\infty)$ representing the time, position and lifetime of the the cluster with rate,

$$\begin{aligned}\tilde{\Lambda}^\varepsilon(dt, dx, d\tau) &= \frac{1}{\varepsilon} dt \psi(x) dx \frac{2\varepsilon}{\tau^2} e^{-\frac{2\varepsilon}{\tau}} d\tau \\ &= \frac{2}{\tau^2} e^{-\frac{2\varepsilon}{\tau}} \psi(x) dt dx d\tau.\end{aligned}$$

We will now thin $\tilde{\Xi}^{\varepsilon}$ into the clusters that live longer than time 1 and clusters that die before time 1. This gives rise to two new independent PPP's, $\tilde{\Xi}^{1,\varepsilon}$ and $\tilde{\Xi}^{2,\varepsilon}$ with rates $\tilde{\Lambda}^{1,\varepsilon}$ and $\tilde{\Lambda}^{2,\varepsilon}$ respectively given by,

$$\tilde{\Lambda}^{1,\varepsilon}(dt, dx, d\tau) = \frac{2}{\tau^2} e^{-\frac{2\varepsilon}{\tau}} \psi(x) 1(\tau > 1) dt dx d\tau,$$
$$\tilde{\Lambda}^{2,\varepsilon}(dt, dx, d\tau) = \frac{2}{\tau^2} e^{-\frac{2\varepsilon}{\tau}} \psi(x) 1(\tau \leq 1) dt dx d\tau.$$

We now suppose $\varepsilon > 0$ is fixed, and $(t_i, x_i, \tau_i) \sim \tilde{\Xi}^{1,\varepsilon}$ and $(s_j, y_j, \sigma_j) \sim \tilde{\Xi}^{2,\varepsilon}$. Let $\{U^i\}$ be independent SBM with initial law $\phi_\varepsilon^{x_i}(z)dz$ originating at (t_i, x_i) conditioned to live greater than 1 unit time. Similarly, let $\{V^j\}$ be independent SBM with initial law $\phi_\varepsilon^{y_j}(z)dz$ originating at (s_j, y_j) conditioned to die by time 1. We will assume without loss of generality that clusters are generated in chronological order, i.e $t_i < t_{i+1}$ and $s_j \leq s_{j+1}$. Note that all the terms defined depend on ε, however the ε was suppressed for notational convenience.

The plan of attack is as follows. Let (t_1, x_1) denote the time and position of the first SBM cluster U^1 generated by $\tilde{\Xi}^{1,\varepsilon}$.

1. We will show for all $\alpha > 0$ small enough, that there is a $p_\alpha > 0$ independent of ε such that, $t_1 \leq \alpha$ and U^1 will contribute to X^ε with probability atleast p_α. This will imply that for all $t > 0$

$$X_t^\varepsilon(1) \geq 1(t \geq t_1) U_{t-t_1}^1(1), \tag{2.71}$$

with probability atleast p_α.

2. We will then show that for all $t > 0$ there is a $q_t > 0$ independent of ε, such that that $U_t^1(1) > 1$ with probability at least q_t.

3. We will use steps 1 and 2 along with weak convergence to show that

$$\mathbb{P}(X_2(1) > 1) > 0.$$

2.7.1 Contribution of long living populations to X^ε

Our goal in this section will be to complete step 1 of the program laid out above. We will show that U^1 will both be generated before time α and will contribute to X^ε with probability at least $p_\alpha > 0$ independent of ε. First let us analyse the time that U^1 is generated.

Lemma 2.7.2. *(a) t_1 is $(\mathcal{F}_t^\varepsilon)$-stopping time.*

(b) t_1 is exponentially distributed with rate

$$\lambda_\varepsilon \equiv \frac{1-e^{-2\varepsilon}}{\varepsilon}.$$

(c) For all $\alpha > 0$ and ε small enough,

$$\mathbb{P}(t_1 \leq \alpha) \geq 1 - e^{-\alpha}.$$

Proof. (a) Recall $\tilde{\Xi}_t^{1,\varepsilon} \sim \mathrm{Poi}(\tilde{\Lambda}_t^{1,\varepsilon})$ is the total number of points generated by $\tilde{\Xi}^{1,\varepsilon}$ up to and including time t. So

$$\{t_1 \leq t\} = \{\tilde{\Xi}_t^{1,\varepsilon} > 0\} \in \mathcal{F}_t^\varepsilon.$$

(b) Let us compute the distribution of t_1.

$$\begin{aligned}
\mathbb{P}(t_1 \leq t) &= \mathbb{P}(\tilde{\Xi}_t^{1,\varepsilon} > 0) \\
&= 1 - \mathbb{P}(\tilde{\Xi}_t^{1,\varepsilon} = 0) \\
&= 1 - \exp\left(-\tilde{\Lambda}_t^{1,\varepsilon}\right) \\
&= 1 - \exp\left(-\int_0^t \int_{-\infty}^\infty \int_0^\infty \frac{2}{\tau^2} e^{-\frac{2\varepsilon}{\tau}} \psi(x) 1(\tau > 1) ds dx d\tau\right) \\
&= 1 - \exp\left(-t \int_1^\infty \frac{2}{\tau^2} e^{-\frac{2\varepsilon}{\tau}} d\tau\right) \\
&= 1 - \exp\left(-t \frac{1-e^{-2\varepsilon}}{\varepsilon}\right) \\
&= 1 - \exp\left(-t\lambda_\varepsilon\right). \qquad (2.72)
\end{aligned}$$

This proves part (b).

(c) Since $\lambda_\varepsilon \to 2$ as $\varepsilon \to 0$, we have for ε small enough, $\lambda_\varepsilon \geq 1$. Thus by (b),

$$\mathbb{P}(t_1 \leq \alpha) = 1 - e^{-\alpha\lambda_\varepsilon} \geq 1 - e^{-\alpha} > 0.$$

□

For all $0 \leq s_j \leq t_1 \leq \alpha$, we want the corresponding clusters V^j to have no mass at (t_1, x_1). In order for this to happen we need to get some control on the support of V^j. The following theorem on the modulus of continuity, gives us an estimate on the support of a SBM.

Theorem 2.7.3 (Modulus of continuity)**.** *(Perkins, 2002, §III.1) Let X be a SBM with initial law X_0, then for all $0 < p < \frac{1}{2}$*

(a) There is a $\delta(\omega) > 0$ such that for $0 < t < \delta$

$$\mathrm{supp}(X_t) \subset \{x : \exists z \in \mathrm{supp}(X_0), |x - z| \leq t^p, \}.$$

(b) There is some ρ and K such that for all $\alpha > 0$,

$$\mathbb{P}(\delta \leq \alpha) \leq K X_0(1) \alpha^\rho.$$

Let δ_j denote the time where the modulus of continuity is satisfied for SBM cluster V^j as described in Theorem 2.7.3. Let A_α^ε denote the event that for all $s_j \leq \alpha$, the corresponding V^j have a modulus of continuity $\delta_j \geq \alpha$. Equivalently,

$$A_\alpha^\varepsilon = \bigcap_{s_j \leq \alpha} \{\delta_j \geq \alpha\}.$$

We will now find an estimate on A_α^ε occurring.

Lemma 2.7.4. *There is an $K, \rho > 0$ such that*

$$\mathbb{P}(A_\alpha^\varepsilon) > e^{-K\alpha^{\rho+1}}.$$

Proof. Since δ_j only depends on the initial mass, and V^j are independent clusters with mass ε, we have δ_j are iid random variables. We then define $p \equiv \mathbb{P}(\delta_j \geq \alpha)$, and as before, let $\tilde{\Xi}^{2,\varepsilon}_\alpha \sim \text{Poi}(\tilde{\Lambda}^{2,\varepsilon}_\alpha)$ denote the the number of points generated by $\tilde{\Xi}^{2,\varepsilon}$ until time α. To simplify notation, let $\lambda \equiv \tilde{\Lambda}^{2,\varepsilon}_\alpha$ be the average number of points generated by $\tilde{\Xi}^{2,\varepsilon}$ by time α. We now compute $\mathbb{P}(A^\varepsilon_\alpha)$.

$$\begin{aligned}
\mathbb{P}(A^\varepsilon_\alpha) &= \mathbb{P}\left(\bigcap_{s_j \leq \alpha} \delta_j \geq \alpha\right) \\
&= \sum_{k=0}^{\infty} \mathbb{P}\left(\bigcap_{s_j \leq \alpha} \delta_j \geq \alpha \middle| \tilde{\Xi}^{2,\varepsilon}_\alpha = k\right) \mathbb{P}(\tilde{\Xi}^{2,\varepsilon}_\alpha = k) \\
&= \sum_{k=0}^{\infty} \prod_{j}^{k} \mathbb{P}(\delta_j \geq \alpha)\mathbb{P}(\tilde{\Xi}^{2,\varepsilon}_\alpha = k) \\
&= \sum_{k=0}^{\infty} p^k e^{-\lambda} \frac{\lambda^k}{k!} \\
&= e^{-\lambda} e^{p\lambda} \\
&= e^{-\lambda(1-p)}.
\end{aligned} \tag{2.73}$$

We now compute λ,

$$\lambda = \tilde{\Lambda}^{2,\varepsilon}_\alpha = \int_0^\alpha \int_{-\infty}^{\infty} \int_0^1 \frac{2}{\tau^2} e^{-\frac{2\varepsilon}{\tau}} \psi(x) d\tau dx dt = \frac{\alpha}{\varepsilon} e^{-2\varepsilon}. \tag{2.74}$$

Also by Theorem 2.7.3,

$$1 - p = 1 - \mathbb{P}(\delta_j \geq \alpha) = \mathbb{P}(\delta < \alpha) \leq K\varepsilon\alpha^\rho \tag{2.75}$$

Combining (2.73) with (2.74) and (2.75), we get

$$\begin{aligned}
P(A^\varepsilon_\alpha) &= \exp\left(-\lambda(1-p)\right) \\
&\geq \exp\left(-\frac{\alpha}{\varepsilon} e^{-\varepsilon} K\varepsilon\alpha^\rho\right) \\
&= \exp\left(-K\alpha^{\rho+1} e^{-\varepsilon}\right) \\
&\geq \exp\left(-K\alpha^{\rho+1}\right).
\end{aligned}$$

□

We now condition on the event $B^\varepsilon_\alpha = \{t_1 \le \alpha\} \cap A^\varepsilon_\alpha$, ie, the event where U^1 is born by time α and all the V^j born before time α satisfy $\delta_j \ge \alpha$. Since $\{t_1 \le \alpha\}$ only depends on the $\tilde{\Xi}^{1,\varepsilon}$ and A^ε_α only depends on V^j, which are both independent from each other, we have $\{t_1 \le \alpha\}$ and A^ε_α are independent events. Therefore by Lemma 2.7.2 and 2.7.4, we have for all ε sufficiently small,

$$\mathbb{P}(B^\varepsilon_\alpha) = \mathbb{P}(t_1 \le \alpha)\mathbb{P}(A^\varepsilon_\alpha) \ge (1 - e^{-\alpha})e^{-K\alpha^{\rho+1}} \equiv \gamma_\alpha > 0. \tag{2.76}$$

Before we proceed, it will be useful to define the space-time graph of a V^j.

Definition 2.7.5. *Let X be a M_F-valued process, let $G(X)$ be the* ***space-time graph*** *of X defined by,*

$$G(X) = \{(x,t) | x \in \text{supp}(X_t)\}.$$

Let G^j denote the space-time graph of the shifted process $1(s_j \le t)V^j_{t-s_j}$, which just ends up being the $G(V^j)$ shifted up by s_j.

$$G^j \equiv G\left(1(s_j \le \cdot)V^j_{\cdot - s_j}\right) = G(V^j) + (s_j, 0).$$

If C^ε is the event that U^1 will contribute to X^ε. Then this will occur if $(t_1, x_1) \notin G^j$ for all j. Therefore

$$C^\varepsilon \supset \bigcap_j \{(t_1, x_1) \notin G^j\}. \tag{2.77}$$

We now estimate the probability of U^1 contributing the sum given B^ε_α has occurred.

Proposition 2.7.6. *For all $\alpha > 0$ we have for all ε sufficiently small,*

$$\mathbb{P}\left(C^\varepsilon | B^\varepsilon_\alpha\right) \ge \exp(-32\|\psi\|_\infty \alpha^{\frac{1}{4}}) > 0. \tag{2.78}$$

Proof. We begin by noting that by translation, we can assume without loss of generality that $(t_1, x_1) = (0,0)$. Suppose that a cluster is created at (s_j, y_j) with lifetime τ_j and $-\alpha \le s_j \le 0$. We want to estimate the probability that $(s_j, y_j) \notin G^j$. Since we are only interested in $-\alpha \leq s_j \leq 0$, we restrict our space-time graphs to $[-\alpha, 0] \times \mathbb{R}$. The modulus of continuity tell us that for $p = \frac{1}{4} < \frac{1}{2}$, we have on B_α^ε,

$$G^j|_{[-\alpha,0]\times\mathbb{R}} \subset \left\{ (s,y) \Big| \begin{array}{c} s_j \leq s \leq 0,\ \exists z \in [y_j - \sqrt{\varepsilon}, y_j + \sqrt{\varepsilon}], \\ |y - z| \leq |s - s_j|^{\frac{1}{4}} \end{array} \right\} \tag{2.79}$$

Also (2.77) implies.

$$\begin{aligned} \mathbb{P}(C^\varepsilon | B_\alpha^\varepsilon) &\geq \mathbb{P}(\forall j, (0,0) \notin G^j | B_\alpha^\varepsilon) \\ &= 1 - \mathbb{P}(\exists j, (0,0) \in G^j | B_\alpha^\varepsilon). \end{aligned} \tag{2.80}$$

We will now use (2.79) to find an upper bound on $\mathbb{P}(\exists j, (0,0) \in G^j | B_\alpha^\varepsilon)$. We have two cases:

Case 1: $|y_j| > \sqrt{\varepsilon}$.

In this case $(s_j, 0) \notin G^j$, since $V_0^j = \phi_\varepsilon^{y_j}(z)dz$ is support on $[y_j - \sqrt{\varepsilon}, y_j + \sqrt{\varepsilon}]$ and doesn't contain 0. Note that for $s > s_j \geq -\alpha$, (2.79) tells us $(s,y) \in G^j$, if for some $z \in [y_j - \sqrt{\varepsilon}, y_j + \sqrt{\varepsilon}]$,

$$|y - z| \leq |s - s_j|^{\frac{1}{4}} \iff s \geq s_j + |y - z|^4. \tag{2.81}$$

This implies if $(0,0) \in G^j$, then

$$0 \geq s_j + |0 - z|^4 \geq s_j + \left(|y_j| - \sqrt{\varepsilon}\right)^4 \geq s_j \geq -\alpha. \tag{2.82}$$

By rearranging (2.82), we get s_j and y_j must satisfy

$$-\alpha \leq s_j \leq -\left(|y_j| - \sqrt{\varepsilon}\right)^4, \tag{2.83}$$

and,

$$\sqrt{\varepsilon} < |y_j| \leq \alpha^{\frac{1}{4}} + \sqrt{\varepsilon}, \tag{2.84}$$

respectively. Also for $(0,0) \in G^j$ to be true, the cluster needs to live long enough to hit the origin, so

$$|s_j| \leq \tau_j. \tag{2.85}$$

Let A be the set of (s_j, y_j, τ_j) such that (2.83), (2.85), and (2.84) hold. So if $\tilde{\Xi}^{2,\varepsilon}(A) > 0$ then $(0,0) \in G^j$ for some j.

Case 2: $|y_j| \leq \sqrt{\varepsilon}$.

In this case the $V_0^j = \phi_\varepsilon^{y_j}(z)dz$ which occupies mass at 0. Having $(0,0) \in S^j$ implies that V^j lives at least $|s_j|$, which implies there is cluster such that $\tau \geq |t_j|$. If B is the set of (s_j, y_j, τ_j) such that $\tau_j > |s_j|$, $-\alpha \leq s_j \leq 0$, and $|y_j| \leq \sqrt{\varepsilon}$, then $\tilde{\Xi}^{2,\varepsilon}(B) > 0$ implies $(0,0) \in G^j$ for some j.

Combining case 1 and 2, we have shown that

$$\{\exists j, (0,0) \in G^j\} \subset \{\tilde{\Xi}^{2,\varepsilon}(A) > 0\} \cup \{\tilde{\Xi}^{2,\varepsilon}(B) > 0\} = \{\tilde{\Xi}^{2,\varepsilon}(A \cup B) > 0\}.$$

Since $A \cap B = \emptyset$, $\{\tilde{\Xi}^{2,\varepsilon}(A) > 0\}$ and $\{\tilde{\Xi}^{2,\varepsilon}(B) > 0\}$ are independent events. This implies,

$$\begin{aligned}
\mathbb{P}(\exists j, (0,0) \in S^j | B_\alpha^\varepsilon) &\leq \mathbb{P}(\tilde{\Xi}^{2,\varepsilon}(A \cup B) > 0 | B_\alpha^\varepsilon) \\
&= \mathbb{P}(\tilde{\Xi}^{2,\varepsilon}(A \cup B) > 0) \\
&= 1 - \mathbb{P}(\tilde{\Xi}^{2,\varepsilon}(A \cup B) = 0) \\
&= 1 - \exp(-\tilde{\Lambda}^{2,\varepsilon}(A \cup B)) \\
&= 1 - \exp(-\tilde{\Lambda}^{2,\varepsilon}(A) - \tilde{\Lambda}^{2,\varepsilon}(B)). \qquad (2.86)
\end{aligned}$$

The first inequality used B_α^ε depends on δ_j and t_1, both of which are independent of $\tilde{\Xi}^{2,\varepsilon}$.

Whe last line used the fact that A and B are disjoint. So now lets compute $\tilde{\Lambda}_2^{\varepsilon}(A)$, $\tilde{\Lambda}_2^{\varepsilon}(B)$.

$$
\begin{aligned}
\tilde{\Lambda}_2^{\varepsilon}(A) = & \int_{\sqrt{\varepsilon}<|y|\leq\alpha^{\frac{1}{4}}+\sqrt{\varepsilon}} \int_{-\alpha}^{-\left(|y|-\sqrt{\varepsilon}\right)^4} \int_{|s|}^{1} \frac{2}{\tau^2} e^{-\frac{2\varepsilon}{\tau}} \psi(y) d\tau ds dy \\
\leq & \; 2\|\psi\|_\infty \int_{\sqrt{\varepsilon}<|y|\leq\alpha^{\frac{1}{4}}+\sqrt{\varepsilon}} \int_{-\alpha}^{-\left(|y|-\sqrt{\varepsilon}\right)^4} \int_{|s|}^{\infty} \frac{1}{\tau^2} d\tau ds dy \\
= & \; 2\|\psi\|_\infty \int_{\sqrt{\varepsilon}<|y|\leq\alpha^{\frac{1}{4}}+\sqrt{\varepsilon}} \int_{-\alpha}^{-\left(|y|-\sqrt{\varepsilon}\right)^4} \frac{1}{|s|} ds dy \\
= & \; 2\|\psi\|_\infty \int_{\sqrt{\varepsilon}<|y|\leq\alpha^{\frac{1}{4}}+\sqrt{\varepsilon}} \log\left|\frac{\alpha}{(|y|-\sqrt{\varepsilon})^4}\right| dy \\
= & \; 4\|\psi\|_\infty \int_{\sqrt{\varepsilon}}^{\alpha^{\frac{1}{4}}+\sqrt{\varepsilon}} \log\left|\frac{\alpha}{(y-\sqrt{\varepsilon})^4}\right| dy \\
= & \; 16\|\psi\|_\infty \int_{\sqrt{\varepsilon}}^{\alpha^{\frac{1}{4}}+\sqrt{\varepsilon}} \log\left|\frac{\alpha^{\frac{1}{4}}}{y-\sqrt{\varepsilon}}\right| dy \\
= & \; 16\|\psi\|_\infty \alpha^{\frac{1}{4}} \int_0^1 \log\left|\frac{1}{u}\right| du \\
= & \; 16\|\psi\|_\infty \alpha^{\frac{1}{4}}.
\end{aligned} \tag{2.87}
$$

In the second last line we used the substitution,

$$
u = \frac{y-\sqrt{\varepsilon}}{\alpha^{\frac{1}{4}}}.
$$

Now let's compute $\tilde{\Lambda}_2^\varepsilon(B)$.

$$\begin{aligned}
\tilde{\Lambda}_2^\varepsilon(B) &= \int_{-\sqrt{\varepsilon}}^{\sqrt{\varepsilon}} \int_{-\alpha}^{0} \int_{|s|}^{1} \frac{2}{\tau^2} e^{-\frac{2\varepsilon}{\tau}} \psi(y) d\tau ds dy \\
&\leq \|\psi\|_\infty \int_{-\sqrt{\varepsilon}}^{\sqrt{\varepsilon}} \int_{-\alpha}^{0} \int_{|s|}^{\infty} \frac{2}{\tau^2} e^{-\frac{2\varepsilon}{\tau}} d\tau ds dy \\
&= \|\psi\|_\infty \int_{-\sqrt{\varepsilon}}^{\sqrt{\varepsilon}} \int_{-\alpha}^{0} \frac{1 - e^{-\frac{2\varepsilon}{|s|}}}{\varepsilon} ds dy \\
&= 2\|\psi\|_\infty \int_0^\alpha \frac{1 - e^{-\frac{2\varepsilon}{s}}}{\sqrt{\varepsilon}} ds \\
&= \frac{2\|\psi\|_\infty}{\sqrt{\varepsilon}} \left[\int_0^\varepsilon 1 - e^{-\frac{2\varepsilon}{s}} ds + \int_\varepsilon^\alpha 1 - e^{-\frac{2\varepsilon}{s}} ds \right] \\
&\leq \frac{2\|\psi\|_\infty}{\sqrt{\varepsilon}} \left[\int_0^\varepsilon 1 dt + \int_\varepsilon^\alpha \frac{2\varepsilon}{s} ds \right] \\
&= \frac{2\|\psi\|_\infty}{\sqrt{\varepsilon}} \left[\varepsilon - 2\varepsilon \log\left(\frac{\varepsilon}{\alpha}\right) \right] \\
&= 2\|\psi\|_\infty \left[\sqrt{\varepsilon} - 2\sqrt{\varepsilon} \log\left(\frac{\varepsilon}{\alpha}\right) \right]. \qquad (2.88)
\end{aligned}$$

Which goes to 0 as $\varepsilon \to 0$. So we can pick ε small enough so that the last line is less than $16\|\psi\|_\infty \alpha^{\frac{1}{4}}$. Combining (2.86), (2.87), and (2.88), we end up with,

$$\mathbb{P}(\exists j, (0,0) \in S^j | B_\alpha^\varepsilon) \leq 1 - \exp(-32\|\psi\|_\infty \alpha^{\frac{1}{4}}). \qquad (2.89)$$

Therefore (2.80), and (2.89) imply,

$$\mathbb{P}\left(C^\varepsilon | B_\alpha^\varepsilon\right) \geq \exp(-32\|\psi\|_\infty \alpha^{\frac{1}{4}}).$$

□

Let $C_\alpha^\varepsilon \equiv C^\varepsilon \cap B_\alpha^\varepsilon$, be the event where U^1 is born before time α, contributes to X^ε, and all V^j born before time alpha satisfy the modulus of continuity for atleast time α after birth.

Remark 2.7.7. Note that C_α^ε depends on t_1 up until time α, $\tilde{\Xi}^{2,\varepsilon}$ upto time α, the V^j born by time α for a time α after their birth. Therefore $C_\alpha^\varepsilon \in \mathcal{F}_{2\alpha}^\varepsilon$.

We end this section by estimating C_α^ε, we will need the following lemma.

Corollary 2.7.8. *For all $\alpha > 0$, there is a $p_\alpha > 0$ such that for all ε small enough,*

$$P(C_\alpha^\varepsilon) \geq p_\alpha > 0,$$

independent of ε.

Proof. By (2.76) and Proposition 2.7.6,

$$\begin{aligned} P(C_\alpha^\varepsilon) &= \mathbb{P}(C^\varepsilon \cap B_\alpha^\varepsilon) \\ &= \mathbb{P}(C^\varepsilon | B_\alpha^\varepsilon)\mathbb{P}(B_\alpha^\varepsilon) \\ &\geq \exp(-32\|\psi\|_\infty \alpha^{\frac{1}{4}})\gamma_\alpha \\ &\equiv p_\alpha \\ &> 0. \end{aligned}$$

□

2.7.2 Proof of Non-triviality of X

Recall if C_α^ε occurs, then U^1 contributes to X^ε and $t_1 \leq \alpha$. By Corollary 2.7.8, we know that, there is a $p_\alpha > 0$ independent of ε such that $\mathbb{P}(C_\alpha^\varepsilon) \geq p_\alpha$. If C_α^ε has occurred, we have for all $t \geq 0$,

$$X_t^\varepsilon(1) \geq 1(t \geq t_1)U_{t-t_1}^1(1). \tag{2.90}$$

The goal for this section is to show that X is non-trivial by showing that $\mathbb{P}(X_2(1) > 1)$ with positive probability. We will do this by estimating $\mathbb{P}(X_2^{\varepsilon_n}(1) > 1)$ and using the definition of weak convergence. Before we do this, we will need to examine U^1 a bit closer.

Recall that U^1 is a SBM with initial law $\phi_\varepsilon^{x_1}(z)dz$ conditioned to live longer then

1. If U is an independent SBM with initial law $\phi_\varepsilon^{x_1}(z)dz$ with lifetime τ, then for $t \geq 1$,

$$\begin{aligned}\mathbb{P}(U_t^1(1) > 1) &= \mathbb{P}(U_t(1) > 1 | \tau > 1) \\ &= \frac{\mathbb{P}(U_t(1) > 1, \tau > 1)}{\mathbb{P}(\tau > 1)} \\ &= \frac{\mathbb{P}(U_t(1) > 1)}{\mathbb{P}(\tau > 1)} \\ &= \frac{\mathbb{P}(U_t(1) > 1)}{1 - e^{-2\varepsilon}}. \end{aligned} \tag{2.91}$$

The second last line used the fact that $t \geq 1$. In order to compute the numerator of (2.91), we will first need to determine the distribution of $U_t(1)$. We will do this by using moment generating functions.

Lemma 2.7.9. *Suppose $Y_1, Y_2, \ldots$ are iid random variables, and N is an independent random variable taking on non-negative integer values with MGF's M_Y and M_N respectively. Then $Z = \sum_{i=1}^N Y_i$ has the moment generating functions*

$$M_Z(s) = M_N(\log M_Y(s)).$$

Proof. Note that $\mathbb{E}[\exp(sZ)|N = n] = M_Y^n(s) = \exp(n \log M_Y(s))$. Therefore by taking conditional expectation,

$$\begin{aligned} M_Z(s) &= \mathbb{E}[\exp(sZ)] \\ &= \sum_{n=0}^{\infty} \mathbb{E}[\exp(sZ)|N = n]\mathbb{P}(N = n) \\ &= \sum_{n=0}^{\infty} \exp(n \log M_Y(s))\mathbb{P}(N = n) \\ &= M_N(\log M_Y(s)). \end{aligned}$$

□

We will now find the distribution of $U_t(1)$.

Lemma 2.7.10. *For all $t > 0$, we have*

$$U_t(1) \overset{d}{=} \sum_{i=1}^{N(t)} Y_i(t).$$

Where $Y_i(t)$ are iid exponential random variables with rate $\frac{2}{t}$, and $N(t)$ is an independent Poisson random variable with mean $\frac{2\varepsilon}{t}$.

Proof. We will do this by showing that both $U_t(1)$ and $Z(t) \equiv \sum_{i=1}^{N(t)} Y_i(t)$ have the same MGF. By analysing the Laplace functional for U, it was shown in [Cite St. Fleur and Knight], that the MGF of $U_t(1)$ is,

$$M_{U_t(1)}(s) = \exp\left(\frac{2\varepsilon s}{2 - ts}\right). \tag{2.92}$$

The MGF of Y_i and N are,

$$M_{Y(t)}(s) = \frac{\frac{2}{t}}{\frac{2}{t} - s} = \frac{2}{2 - ts}, \tag{2.93}$$

$$M_{N(t)}(s) = \exp\left(\frac{2\varepsilon}{t}(e^s - 1)\right). \tag{2.94}$$

Therefore by Lemma 2.7.9 we have,

$$\begin{aligned}
M_{Z(t)}(s) &= M_{N(t)}(\log M_{Y(t)}(s)) \\
&= \exp\left(\frac{2\varepsilon}{t}(M_{Y(t)}(s) - 1)\right) \\
&= \exp\left(\frac{2\varepsilon}{t}\left(\frac{2}{2 - ts} - 1\right)\right) \\
&= \exp\left(\frac{2\varepsilon s}{2 - ts}\right) \\
&= M_{U_t(1)}(s).
\end{aligned}$$

Since $Z(t)$ and $U_t(1)$ have the same MGF, they have the same distribution. □

Finally we compute the probability that $U_t^1(1) > 1$.

Proposition 2.7.11. *For $t \geq 1$,*

$$\mathbb{P}(U_t^1(1) > 1) \geq \frac{e^{-\frac{4}{t}}}{t}.$$

Proof. Continuing our computation from (2.91), we have by Lemma 2.7.10, if $N(t)$ is a Poisson random variable with mean $\frac{2\varepsilon}{t}$ and $Y_i(t)$ are iid exponentially distributed random variables with rate $\frac{2}{t}$, then,

$$\begin{aligned}
\mathbb{P}(U_t^1(1) > 1) &= \frac{\mathbb{P}(U_t(1) > 1)}{1 - e^{-2\varepsilon}} \\
&\geq \frac{\mathbb{P}(U_t(1) > 1)}{2\varepsilon} \\
&= \frac{\mathbb{P}\left(\sum_{i=1}^{N(t)} Y_i(t) > 1\right)}{2\varepsilon} \\
&\geq \frac{\mathbb{P}\left(\sum_{i=1}^{N(t)} Y_i(t) > 1, N(t) = 1\right)}{2\varepsilon} \\
&= \frac{\mathbb{P}\left(\sum_{i=1}^{N(t)} Y_i(t) > 1 | N(t) = 1\right) \mathbb{P}(N(t) = 1)}{2\varepsilon} \\
&= \frac{\mathbb{P}\left(Y_1(t) > 1\right) \mathbb{P}(N(t) = 1)}{2\varepsilon} \\
&= \frac{e^{-\frac{2}{t}} \frac{2\varepsilon}{t} e^{-\frac{2\varepsilon}{t}}}{2\varepsilon} \\
&= \frac{1}{t} e^{-\frac{4\varepsilon}{t}}
\end{aligned}$$

Since we are interested in ε small, we assume $\varepsilon \leq 1$. Therefore,

$$\mathbb{P}(U_t^1(1) > 1) \geq \frac{e^{-\frac{4}{t}}}{t}.$$

□

Finally, this estimate in enough to conclude that X is not trivial.

Theorem 2.7.12. *The process X defined as the solution to the SPDE* (2.70)*, is not identically zero.*

Proof. To show non-triviality of X, we will show that

$$\mathbb{P}(X_2(1) > 1) > 0. \tag{2.95}$$

Let $\varepsilon > 0$ be fixed, and $\alpha < \frac{1}{2}$. We will estimate $\mathbb{P}(X_2^\varepsilon(1) > 1)$ by conditioning on $\mathcal{F}_{t_1+1}^\varepsilon$, and using (2.90).

$$\begin{aligned}\mathbb{P}(X_2^\varepsilon(1) > 1) &\geq \mathbb{P}(X_2^\varepsilon(1) > 1, C_\alpha^\varepsilon) \\ &\geq \mathbb{P}(U_{2-t_1}^1(1) > 1, C_\alpha^\varepsilon) \\ &= \mathbb{E}(\mathbb{P}(U_{2-t_1}^1(1) > 1, C_\alpha^\varepsilon | \mathcal{F}_{t_1+1}^\varepsilon)) \\ &= \mathbb{E}(1(C_\alpha^\varepsilon)\mathbb{P}(U_{2-t_1}^1(1) > 1 | \mathcal{F}_{t_1+1}^\varepsilon)) \end{aligned} \tag{2.96}$$

The last line is true by remark 2.7.7, since $C_\alpha^\varepsilon \in \mathcal{F}_{2\alpha}^\varepsilon \subset \mathcal{F}_{t_1+1}^\varepsilon$. Lemma 2.7.2 tells us that $t_1 + 1$ is a $(\mathcal{F}_t^\varepsilon)$-stopping time. So by the strong Markov property,

$$1(C_\alpha^\varepsilon)\mathbb{P}(U_{2-t_1}^1(1) > 1 | \mathcal{F}_{t_1+1}^\varepsilon) = 1(C_\alpha^\varepsilon)\mathbb{P}_{U_1^1(1)}(U_{1-t_1}^1(1) > 1) \tag{2.97}$$

and noting that C_α^ε has occurred. Since the total mass of a SBM is a Feller branching diffusion process (Perkins, 2002, §II.5), applying the stochastic monotonicity in initial conditions for the Feller branching diffusion to (2.97) gives us,

$$\begin{aligned}1(C_\alpha^\varepsilon)\mathbb{P}(U_{2-t_1}^1(1) > 1 | \mathcal{F}_{t_1+1}^\varepsilon) &\geq 1(C_\alpha^\varepsilon)1(U_1^1(1) > 1)\mathbb{P}_{U_1^1(1)}(U_{1-t_1}^1(1) > 1) \\ &\geq 1(C_\alpha^\varepsilon)1(U_1^1(1) > 1)\mathbb{P}_1(U_{1-t_1}^1(1) > 1) \\ &\geq 1(C_\alpha^\varepsilon)1(U_1^1(1) > 1) \inf_{\frac{1}{2} \leq t \leq 1} \mathbb{P}_1(U_t^1(1) > 1). \end{aligned} \tag{2.98}$$

The last line was because $t_1 \leq \alpha < \frac{1}{2}$. In (Knight, 1981, p. 100), it was shown that the Markov transition kernel for a Feller branching diffusion is,

$$p_{U^1(1)}(t, x, y) = \frac{\sqrt{xy}}{t} \exp\left(-\frac{2(x+y)}{t}\right) I_1\left(4\frac{\sqrt{xy}}{t}\right),$$

with respect to the measure $m(dy) = 2y^{-1}dy$. Where I_1 is the modified Bessel function of the first kind. So

$$\mathbb{P}_1(U_t^1(1) > 1) = \int_1^\infty p_{U^1(1)}(t, 1, y) dm(y) > 0. \tag{2.99}$$

Since $p_{U^1(1)}$ is continuous is t, (2.99) shows $\mathbb{P}_1(U_t^1(1) > 1)$ is also continuous in t for $\frac{1}{2} \leq t \leq 1$. Thus by extreme value theorem,

$$q \equiv \inf_{\frac{1}{2} \leq t \leq 1} \mathbb{P}_1(U_t^1(1) > 1) > 0. \tag{2.100}$$

(2.96), (2.98), and (2.100) together imply,

$$\mathbb{P}(X_2^\varepsilon(1) > 1) \geq q\mathbb{P}(C_\alpha, U_1^1(1) > 1). \tag{2.101}$$

Note that C_α^ε depends on $\tilde{\Xi}^{2,\varepsilon}$, V^j, and (t_1, x_1). Whereas, U^1 is independent of $\tilde{\Xi}^{2,\varepsilon}$, V^j, and the total mass $U_1^1(1)$ is independent of (t_1, x_1). So $U_1^1(1) > 1$ and C_α^ε are independent events and (2.101) becomes,

$$\mathbb{P}(X_2^\varepsilon(1) > 1) \geq q\mathbb{P}(C_\alpha)\mathbb{P}(U_1^1(1) > 1) \geq qp_\alpha e^{-4}. \tag{2.102}$$

The last inequality was by Corollary 2.7.8, and Proposition 2.7.11 for $t = 1$.

Finally, let $\varepsilon_n \to 0$ be a sequence such that $X^{\varepsilon_n}(1)$ converges weakly to X as $n \to \infty$. This implies that $X_2^{\varepsilon_n}(1)$ converges to $X_2(1)$ weakly as random variables. Therefore

$$\mathbb{P}(X_2(1) > 1) = \lim_{n\to\infty} \mathbb{P}(X_2^{\varepsilon_n}(1) > 1) \geq qp_\alpha e^{-4} > 0.$$

□

2.8 Summary

A lot has happened in Chapter 2, we will review what we have accomplished. We began in Section 2.2, where we defined the M_F-valued process X^ε as the sum of independent SBM clusters generated by a PPP Ξ^ε, that contribute to X^ε only when born at unoccupied sites. We showed in Theorem 2.2.2, that X^ε solved the following SPDE,

$$X_t^\varepsilon(\phi) = \int_0^t X_s^\varepsilon\left(\frac{\Delta\phi}{2}\right) ds + M_t^\varepsilon(\phi) + A_t^\varepsilon(\phi).$$

Where $\phi \in C_b^\infty$, $M^\varepsilon(\phi)$ is a continuous $(\mathcal{F}_t^\varepsilon)$-martingale with quadratic variation $\int_0^t X_s^\varepsilon(\phi^2)ds$, and

$$A_t^\varepsilon(\phi) = \int_0^t \int \langle\phi, \phi_\varepsilon^x\rangle 1(u_s^\varepsilon(x) = 0)\Xi^\varepsilon(ds, dx),$$

adds mass of size ε at unoccupied sites governed by Ξ^ε.

We proceeded to Section 2.3 where we show that the family $\{X^{\varepsilon_n}\}$ and $\{A^{\varepsilon_n}\}$ were C-relatively compact in $D(M_F)$ for $\varepsilon_n \to 0$. This was a long process to but we succeeded in proving Theorem 2.3.4. We did this by showing that for all $\phi \in C_c^\infty$, $\{X^{\varepsilon_n}(\phi)\}$, and $\{A^{\varepsilon_n}(\phi)\}$ were C-relatively compact in $D(\mathbb{R})$ and $\{X^{\varepsilon_n}\}$, and $\{A^{\varepsilon_n}\}$ satisfied the compact containment condition in Proposition 2.3.11 and Propostion 2.3.14 respectively. This allowed us to apply Jakubowski's theorem to get the existence of continuous M_F-valued processes X and A as weak limit points of X^{ε_n} and A^{ε_n}. In Section 2.4 we made X more concrete by showing that it solves the SPDE,

$$X_t(\phi) = \int_0^t X_s \left(\frac{\Delta\phi}{2}\right) ds + M_t(\phi) + A_t(\phi). \tag{2.103}$$

Where $\phi \in C_b^\infty$, and $M(\phi)$ is a continuous $(\mathcal{F}_t)$-martingale with quadratic variation $\langle M(\phi)\rangle = \int_0^t X_s(\phi^2)ds$.

The construction of X^ε led to the existence of a process with sample paths in $D(C_0)$ such that

$$X_t^\varepsilon(dx) = u_t^\varepsilon(x)dx$$

for all t. In Section 2.5, we showed u^{ε_n} are C-relatively compact in $D(C_0)$ via Theorem 2.5.1 and then Corollary 2.5.2 deduced that each weak limit point u a continuous C_0-valued process such that

$$X_t(dx) = u_t(x)dx.$$

In Section 2.6, we used the fact that u is a weakly limit point of u^ε in combination with Skorohod's theorem to show that for all t,

$$A_t \ll 1(u_t(x) = 0)dx,$$

as stated in Theorem 2.6.1. This shows that the immigration term A is only activated at sites of zero occupancy by the population X.

It remained to justify that our process X was not the trivial process. The sole purpose of Section 2.7 to rule out this possibility and was done in a few steps. We first showed that with some positive probability independent of ε, there is a SBM cluster U^1 of initial mass ε born by time α, and that contributes to X^ε. We then show again, with positive probability independent of ε, U^1 grows to at least to size 1 by time 1. An application of the strong Markov property, and weak convergence allowed us to deduce that

$$\mathbb{P}(X_2(1) > 1) > 0,$$

and as shown in Theorem 2.7.12.

In conclusion, we constructed a non-trivial process X that solves the SPDE (2.103), and has a continuous immigration A, that only contributes when the population X has zero occupancy.

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