Group structure on spheres and the Hopf fibration

Spheres of spheres over spheres

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Outline

1. Groups and Spheres
2. Hopf Fibration
3. Quantum mechanics and the qubit system
Spheres

Definition

We define the \textbf{n-sphere} $S^n$ to be the set of points in $\mathbb{R}^{n+1}$ of unit distance from the origin. I.e,

$$S^n = \{ x \in \mathbb{R}^{n+1} | |x| = 1 \}$$

Example

$$S^0 = \{-1, 1\}$$
$$S^1 = \{ e^{i\theta} \in \mathbb{C} \cong \mathbb{R}^2 | \theta \in [0, \pi) \}$$
$$S^2$$ is the standard sphere in $\mathbb{R}^3$
Spheres

Figure: 1-sphere

Figure: 2-sphere
Groups

Definition

A **group** is a set $G$ with a multiplication defined such that

1. $\exists e \in G$ such that $\forall g \in G$, $eg = ge = g$
2. $\forall g \in G$, $\exists g^{-1}$ such that $gg^{-1} = g^{-1}g = e$
3. The multiplication is associative, as in $\forall g, h, k \in G$, $(gh)k = g(hk)$

Example

1. $S^0$ is finite group $\mathbb{Z}_2$
2. $S^1$ is $U(1)$, the set of 1-dimensional unitary matrices

It is natural to ask, is $S^n$ always a group? If so why, and if not which ones are?
Why are spheres groups

What makes $S^0$ a group is that we can multiply the unit normed elements of $\mathbb{R}$, and the elements of $S^0$ are closed under real multiplication.

Similarly what makes $S^1$ a group is that the elements can be viewed as unit normed elements in $\mathbb{C} \cong \mathbb{R}^2$. The set of unit normed elements are closed under complex multiplication.

Basically $\mathbb{R}$ and $\mathbb{C}$ are “nice”.
Normed real division algebras

It turns out the common thread between $\mathbb{R}, \mathbb{C}$ is that they are both normed real division algebras.

Definition

An $n$-dimensional **normed real division algebra** $\mathbb{A}$ satisfies the following

1. $\mathbb{A}$ is a normed real vector space
2. $\mathbb{A}$ is a division ring, that may or may not be associative.
3. The norm respects multiplication, as in $\forall a, b \in \mathbb{A}$ we have $|ab| = |a||b|$
In general, if one has an associative $n$-dimensional normed real division algebra $\mathbb{A}$ then we have a group structure on $S^{n-1} = \{ x \in \mathbb{A} \, | \, |x| = 1 \}$ given by the multiplication of $\mathbb{A}$. 
Conversely, if one has a group structure on $S^{n-1}$, one can construct an associative $n$-dimensional normed real division algebra $\mathbb{A}$, via $a, b \in \mathbb{R}^n$ then

$$ab \equiv |a||b| \left( \frac{a}{|a|} \ast \frac{b}{|b|} \right).$$
So we have translated this problem of finding all the spheres with a group structure to finding all normed real division algebras.

It turns out there are a very limited class of normed real division algebras.
Classification of $\mathbb{A}$

So we have translated this problem of finding all the spheres with a group structure to finding all normed real division algebras.

It turns out there are a very limited class of normed real division algebras.

**Theorem (Hurwitz, 1898)**

*There are only 4 normed real division algebras up to isomorphism. They are denoted by $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ and are of dimension 1, 2, 4, 8 respectively. Where $\mathbb{H}$ are the quaternions and $\mathbb{O}$ are the octonions.*
The intuitive reason as to why there are only 4 is that you lose structure every time dimension increases:

- $\mathbb{R}$ to $\mathbb{C}$ one loses ordering
- $\mathbb{C}$ to $\mathbb{H}$ one loses commutativity
- $\mathbb{H}$ to $\mathbb{O}$ one loses associativity

For dimension greater than 8, you lose too much structure.
So we have that the only spheres that are groups are

\[ S^0 \cong \mathbb{Z}_2 \cong O(1), \]
\[ S^1 \cong U(1), \]
\[ S^3 \cong Sp(1) \cong SU(2) \cong SO(3). \]

\( S^7 \) is almost a group, because it lacks associativity.

They will be crucial to the construction of the Hopf fibrations.
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Fibrations

Definition

Let $E, B, F$ be topological spaces. A **fibre bundle** is denoted by

$$F \hookrightarrow E \xrightarrow{p} B$$

where $p : E \to B$ satisfies,

1. $p^{-1}(b) \cong F$
2. $\forall b \in B$ there is a neighbourhood $U$ of $b$ such that $p^{-1}(U)$ is homeomorphic to $U \times F$ via some homeomorphism $\psi : U \times F \to p^{-1}(U)$.
3. We have $p \circ \psi = \pi$ where $\pi : U \times F \to U$ is the projection from $U \times F$ to $U$.

We say that $E$ is the **total space**, $B$ is the **base**, $F$ is the **fibre** and $E$ is the fibre bundle (or fibration) over $B$ with fibre $F$. 
In other words... \[ F \hookrightarrow E \xrightarrow{p} B \]

Is a fancy way of saying \( E \) **locally** looks like “\( B \times F \)” (with some mild technical conditions).
Given $F \leftrightarrow E \overset{p}{\rightarrow} B$, you can think of $E$ as a family of $F$ parametrized by $B$.

In general for all topological spaces $A, B$, the trivial fibration is

$$B \hookrightarrow A \times B \overset{p}{\rightarrow} A$$

where $p((a, b)) = a$

**NOTE:** A fibration is NOT a cartesian product!
Examples: Cylinder

Let $I$ be a closed interval and $p : I \times S^1, p(t, e^{i\theta}) = e^{i\theta}$

$I \hookrightarrow I \times S^1 \xrightarrow{p} S^1$

Figure: $I \times S^1$ or a cylinder, Source: Wikipedia
Examples: Möbius strip

Let $I$ be a closed interval, $M$ the Möbius strip, and $p$ projects to the central circle $S^1$.

$$I \leftrightarrow M \xrightarrow{p} S^1$$

**Figure**: The Möbius strip, Source: virtualmathmuseum.org
Both $I \times S^1$ and $M$ are fibrations over $S^1$ with fibres $I$, but $I \times S^1 \not\cong M$. 
Real projective space

Definition

The real projective space $\mathbb{RP}^n$ is the set of 1 dimensional real subspaces in $\mathbb{R}^{n+1}$. It is a compact, $n$-dimensional smooth manifold.

Figure: $\mathbb{RP}^1$, Source: Wikipedia
Real projective space

Points in $\mathbb{RP}^n$ are the set of equivalence classes in $\mathbb{R}^{n+1}$ such that

$$x, y \in \mathbb{R}^{n+1}, [x] = [y] \iff x = \lambda y \quad \text{for some} \quad 0 \neq \lambda \in \mathbb{R}.$$  

We can restrict our relation to lines intersecting $S^n$ (by picking the representatives of the equivalents classes of unit norm). So we have the set of points in $\mathbb{RP}^n$ are the set of equivalence classes in $S^n$ such that

$$x, y \in S^n, [x] = [y] \iff x = \lambda y \quad \text{for some} \quad 1 = |\lambda|, \lambda \in \mathbb{R}.$$
Real projective space

Note that $\mathbb{RP}^n$ can be though of as the set of orbits of the group action of $S^0$ on $S^n$ by left multiplication. The action is free because $\lambda x = x \Rightarrow \lambda = 1$. So each orbit (ie. fibre) is isomorphic to $S^0$.

Let $\pi : S^n \to \mathbb{RP}^n, \pi(x) = [x]$ be the quotient map. Then we have $S^n$ is a fibration over $\mathbb{RP}^n$ with fibre

$$\pi^{-1}(x) = \{\lambda x | |\lambda| = 1, \lambda \in \mathbb{R}\} = \{x, -x\} \cong S^0.$$  

So we have constructed:

$$S^0 \hookrightarrow S^n \xrightarrow{\pi} \mathbb{RP}^n$$
The complex projective space $\mathbb{CP}^n$ is the set of 1 dimensional complex subspaces in $\mathbb{C}^{n+1}$. It is a compact, $2n$-dimensional smooth manifold.

Points in $\mathbb{CP}^n$ are the set of equivalence classes in $\mathbb{C}^{n+1}$ such that

$$x, y \in \mathbb{C}^{n+1}, [x] = [y] \iff x = \lambda y \quad \text{for some} \quad 0 \neq \lambda \in \mathbb{C}.$$ 

Since $\mathbb{C}^{n+1} \cong \mathbb{R}^{2n+2}$ we pick restrict our relation to lines intersecting $S^{2n+1}$, as before. So we have the set of points in $\mathbb{CP}^n$ are the set of equivalence classes in $S^{2n+1}$ such that

$$x, y \in S^{2n+1}, [x] = [y] \iff x = \lambda y \quad \text{for some} \quad 1 = |\lambda|, \lambda \in \mathbb{C}.$$
Complex projective space

Note that $\mathbb{CP}^n$ can be though of as the set of orbits of the group action of $S^1$ on $S^{2n+1}$ by left multiplication. The action is free because $\lambda x = x \Rightarrow \lambda = 1$. So each orbit (ie. fibre) is isomorphic to $S^1$.

Let $\pi : S^{2n+1} \to \mathbb{CP}^n$, $\pi(x) = [x]$ be the quotient map. Then we have $S^{2n+1}$ is a fibration over $\mathbb{CP}^n$ with fibre

$$\pi^{-1}(x) = \{\lambda x | |\lambda| = 1, \lambda \in \mathbb{C}\} \cong S^1.$$

So we have constructed:

$$S^1 \hookrightarrow S^{2n+1} \xrightarrow{\pi} \mathbb{CP}^n$$
Definition

The quaternionic projective space $\mathbb{H}\mathbb{P}^n$ is the set of 1 dimensional quaternionic subspaces in $\mathbb{H}^{n+1}$. It is a compact, $4n$-dimensional smooth manifold. One has to be a bit careful with multiplication since $\mathbb{H}$ is not commutative.

After repeating the identical process for $\mathbb{RP}^n$, and $\mathbb{CP}^n$, we have $S^{4n+3}$ is a fibration over $\mathbb{H}\mathbb{P}^n$ with fibre

$$\pi^{-1}(x) = \{ \lambda x | |\lambda| = 1, \lambda \in \mathbb{H} \} \cong S^3.$$ 

So we have constructed:

$$S^3 \hookrightarrow S^{4n+3} \xrightarrow{\pi} \mathbb{H}\mathbb{P}^n.$$
It seems natural to repeat the process with $\mathbb{O}$, however the non-associativity of the octonions makes this difficult. It turns out that you cannot define $\mathbb{O}P^n$ for $n > 2$ and can only form a fibration for $\mathbb{O}P^1$, but not over $\mathbb{O}P^2$.

Repeating the previous process we get the following fibrations.

$$S^7 \hookrightarrow S^{8n+7} \overset{\pi}{\rightarrow} \mathbb{O}P^1$$
The Hopf Fibrations

To summarize we have constructed the following fibrations. These are known as the Hopf fibrations.

\[ S^0 \longrightarrow S^n \quad \stackrel{\pi}{\longrightarrow} \quad \mathbb{RP}^n \]
\[ S^1 \longrightarrow S^{2n+1} \quad \stackrel{\pi}{\longrightarrow} \quad \mathbb{CP}^n \]
\[ S^3 \longrightarrow S^{4n+3} \quad \stackrel{\pi}{\longrightarrow} \quad \mathbb{HP}^n \]
\[ S^7 \longrightarrow S^{8+7} \quad \stackrel{\pi}{\longrightarrow} \quad \mathbb{OP}^1 \]

They are usually stated in the case where \( n = 1 \) to get

\[ S^0 \longrightarrow S^1 \quad \stackrel{\pi}{\longrightarrow} \quad \mathbb{RP}^1 \cong S^1 \]
\[ S^1 \longrightarrow S^3 \quad \stackrel{\pi}{\longrightarrow} \quad \mathbb{CP}^1 \cong S^2 \]
\[ S^3 \longrightarrow S^7 \quad \stackrel{\pi}{\longrightarrow} \quad \mathbb{HP}^1 \cong S^4 \]
\[ S^7 \longrightarrow S^{15} \quad \stackrel{\pi}{\longrightarrow} \quad \mathbb{OP}^1 \cong S^8 \]
Let's now look at $S^1 \hookrightarrow S^3 \xrightarrow{\pi} S^2$. This allows us to visualize $S^3$.

If we apply stereographic projection from $S^3$ to $\mathbb{R}^3 \cup \{\infty\}$, have $\mathbb{R}^3$ is completely filled by disjoint circles and a line (circle through $\infty$). Not only that, but all these circles are pairwise “linked”.
Visualization of the 3 — sphere

Figure: Stereographic projection of $S^3$. Each circle is a fibre of $S^3$. Source: sciencenews.org
Outline

1. Groups and Spheres
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3. Quantum mechanics and the qubit system
In quantum mechanics, we study systems corresponding to separable Hilbert spaces, which are complete inner product spaces, with a countable dense set.

The simplest non-trivial system is $\mathcal{H} = \mathbb{C}^2$ corresponds is the qubit system (or spin $\frac{1}{2}$-system).

Besides being an easy system to introduce to an undergrad quantum class, the qubit system is of great importance in quantum cryptography and quantum computing.
Setup

Definition

- We define \{\ket{0}, \ket{1}\} to be an orthonormal basis of \(\mathbb{C}^2\) and \{\bra{0}, \bra{1}\} to be an orthonormal basis for the dual of \(\mathbb{C}^2\).

- So for a general \(\ket{\psi} \in \mathbb{C}^2\) there are some \(a, b \in \mathbb{C}\) such that \(\ket{\psi} = a \ket{0} + b \ket{1}\). We also have that \(\bra{\psi} = \overline{a} \bra{0} + \overline{b} \bra{1}\) is the dual vector of \(\ket{\psi}\).

- Given \(\ket{\psi} = a \ket{0} + b \ket{1}\) and \(\ket{\varphi} = c \ket{0} + d \ket{1}\), we define the inner product on \(\mathbb{C}^2\) by

\[
\bra{\psi} \varphi \rangle := \overline{ac} + \overline{bd}
\]

- We define the norm on \(\mathbb{C}^2\) to be \(\|\ket{\psi}\| := \sqrt{\bra{\psi} \psi \rangle}\)
A quantum **state** is defined to be a vector $|\psi\rangle = a|0\rangle + b|1\rangle$ such that $||\psi|| = |a|^2 + |b|^2 = 1$.

The set of quantum states can be identified with $(u + iv, x + iy)$ in $\mathbb{C}^2$ such that

$$u^2 + v^2 + x^2 + y^2 = 1.$$ 

Therefore set of quantum states is precisely $S^3$, viewed as a subset of $\mathbb{C}^2$. 
States

In quantum mechanics we don’t particularly care about states, but rather what can be observed by them.

If 2 states, always output the same outcomes when ”observed”, then we want to say these states are equivalent. So we need a way to determine how to measure states, and distinguish them.
Observables

Definition

If $\mathcal{H} = \mathbb{C}^2$ is a separable Hilbert space, then an **observable** is an Hermitian operator $A : \mathcal{H} \rightarrow \mathcal{H}$, such that $A^* = A$.

Since $A$ is Hermitian, it has a real eigenvalues, and a can be decomposed as

$$A = \sum_{\lambda \in \text{Spec}(A)} \lambda P_\lambda$$

Where $P_\lambda$ is the projection onto the eigenspace for $\lambda$. 
Outcomes

Definition

- Given an observable $A = \sum_{\lambda \in Spec(A)} \lambda P_{\lambda}$, the outcomes of $A$ are defined to be the eigenvalues of $A$.

- Given a state $|\psi\rangle$ the probability of observing an outcome $\lambda$ with $|\psi\rangle$ is

  $$Pr_{\lambda}(|\psi\rangle) = \langle \psi | P_{\lambda} | \psi \rangle$$

  i.e. the “percentage” of $|\psi\rangle$ that lies in the $\lambda$ eigenspace.
It is natural to define two states to be equal if they always produce the same probabilities.

It is clear from the definition that for all $|\psi\rangle$

$$\Pr_\lambda(|\psi\rangle) = \Pr_\lambda(e^{i\theta}|\psi\rangle).$$

Therefore we define states to be equal if they differ by some $e^{i\theta}$, which is precisely how we defined $\mathbb{C}P^1$. 
Thus in the qubit system quantum states can be viewed as elements of $\mathbb{CP}^1$.

This allows us to use the Hopf fibration to view the set of states in $S^3$ as fibres of $S^1$ parametrized by $S^2$.

In quantum mechanics this is parametrization is called the **Bloch sphere**. It allows us visualize this non trivial space. Fairly complicated actions can be shown to be rotations on $S^2$. 