GROUP-BAYES ESTIMATION

INVITED TALK AT THE
PNWSG MEETING

AND

EXTENDED VERSION OF THE
INVITED TALK AT THE 2004 SSC
MEETING IN MONTREAL

BOTH TALKS IN HONOR OF JIM ZIDEK
ON HIS 65TH BIRTHDAY

by Constance van Eeden

Montreal, 31 May 2004 and
Vancouver, 8 October 2004

A RESULT OF ALEC CHARRAS

Charras (1979)

PhD thesis, Université de Montréal

X with density $\frac{1}{\lambda} f(x/\lambda) \quad x > 0$

$\lambda \geq a \quad a > 0$ known

$\widehat{\lambda}_\rho = \max(\rho X, a)$, $\rho > 0$ known,

an estimator of λ

Conditions on f for the inadmissibility of

$\widehat{\lambda}_\rho$ for squared error loss

Example:

$f(x) = e^{-x} \quad x > 0$

Another example of Charras' result

$X = |Y|$ with Y logistic

$$\mathcal{E}Y = 0$$

$$\sigma(Y) = \lambda$$

Charras and van Eeden (1994)

SUMMARY

1) Group-Bayes inference;

2) Group-Bayes estimation of an exponential mean;

3) Some of our results

See van Eeden and Zidek 1994a and 1994b

GROUP-BAYES INFERENCE

Group of decision makers (DMs).
For instance
a jury or a committee.

data
model(s)
prior(s)
loss function(s)

How to get to a compromise decision?

δ_i the Bayes decision rule of DM$_i$, $i = 1, \ldots, k$

$r_i(\delta)$ DM$_i$'s Bayes risk of a rule δ.

Now define, in analogy to Wald,

a rule δ is group-Bayes inadmissible

if there exists a rule δ_o with

$$r_i(\delta_o) \leq r_i(\delta) \quad \text{for all } i = 1, \ldots, k$$

$$r_i(\delta_o) < r_i(\delta) \quad \text{for some } i \in \{1, \ldots, k\}.$$

Also:

δ_o is group-Bayes minimax if it minimzes

$$\max_{1 \leq i \leq k} r_i(\delta)$$

GROUP-BAYES ESTIMATION OF THE EXPONENTIAL MEAN

$X_1, \ldots, X_n$ i.i.d. $\frac{1}{\lambda} e^{-x/\lambda} I(x > 0)$

The sufficient statistic $T = \sum_{i=1}^{n} X_i$ has density:

$$\frac{1}{\lambda^n \Gamma(n)} t^{n-1} e^{-t/\lambda} \quad t > 0$$

Conjugate priors for λ:

$$\pi_{\alpha,\beta}(\lambda) \propto \lambda^{-\alpha} e^{-\beta/\lambda}$$

with $\alpha > 3$ the same for all DMs and $\beta > 0$.

Lindley's (1976) conjugate utility function:

$$u_{\alpha,\,\beta}(\hat{\lambda}, \lambda) = \gamma_o - \gamma(\hat{\lambda} - \lambda)^2 \quad \gamma > 0$$

Before I go on to tell you about some of the
results Jim and I have on this problem
I want to tell you how and when we got
started on it.

We need to go back to the fall of 1991.
I was spending the semester at UBC.
Jim and I had been talking about
working together
and he proposed a problem which would
combine

his interest in Group-Bayes inference

with my interest in

restricted-parameter-space estimation

He wrote his proposal
on the blackboard in my office

Did not want to lose what he had written

Took some pictures of it

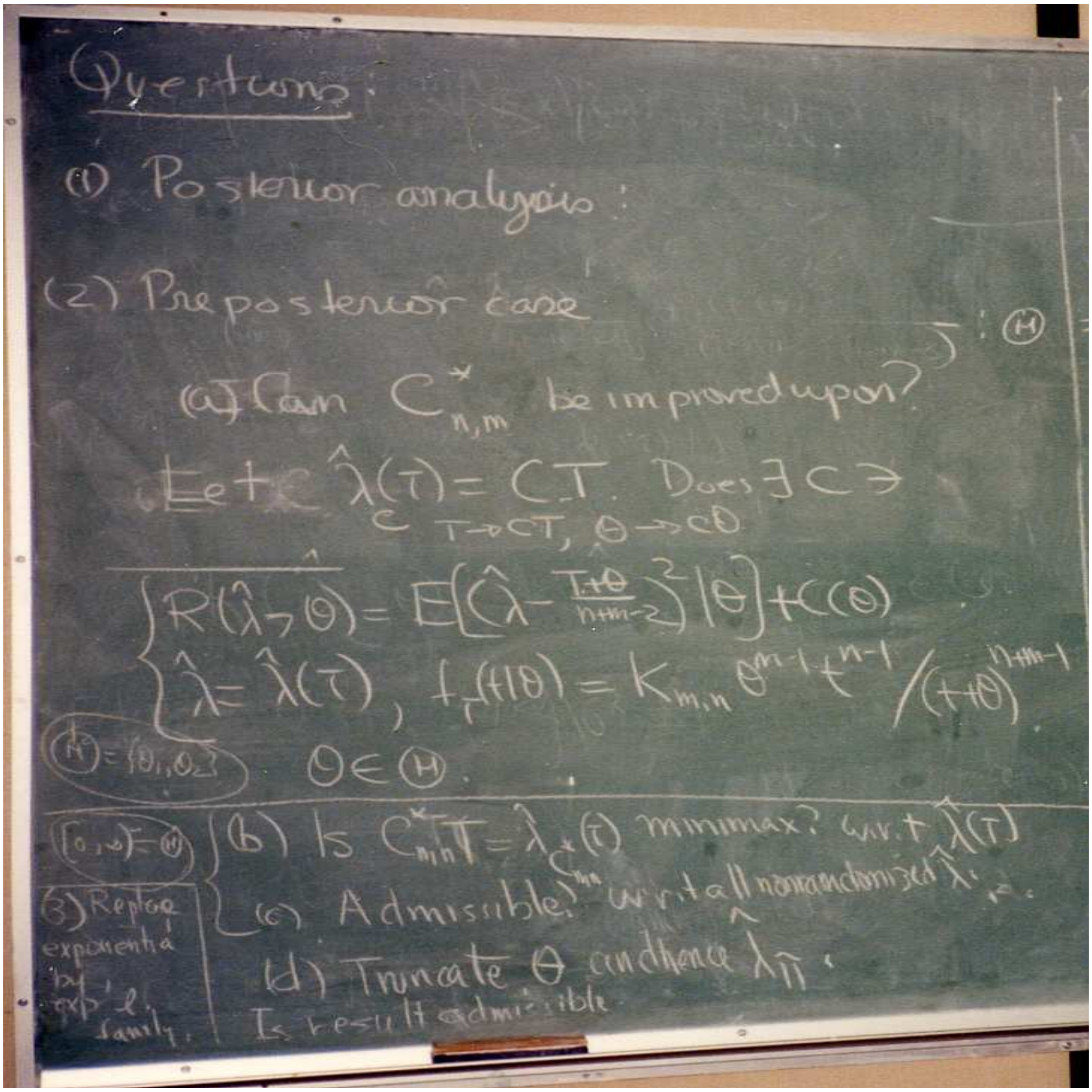
Saved them in a safe place

Found that safe place again

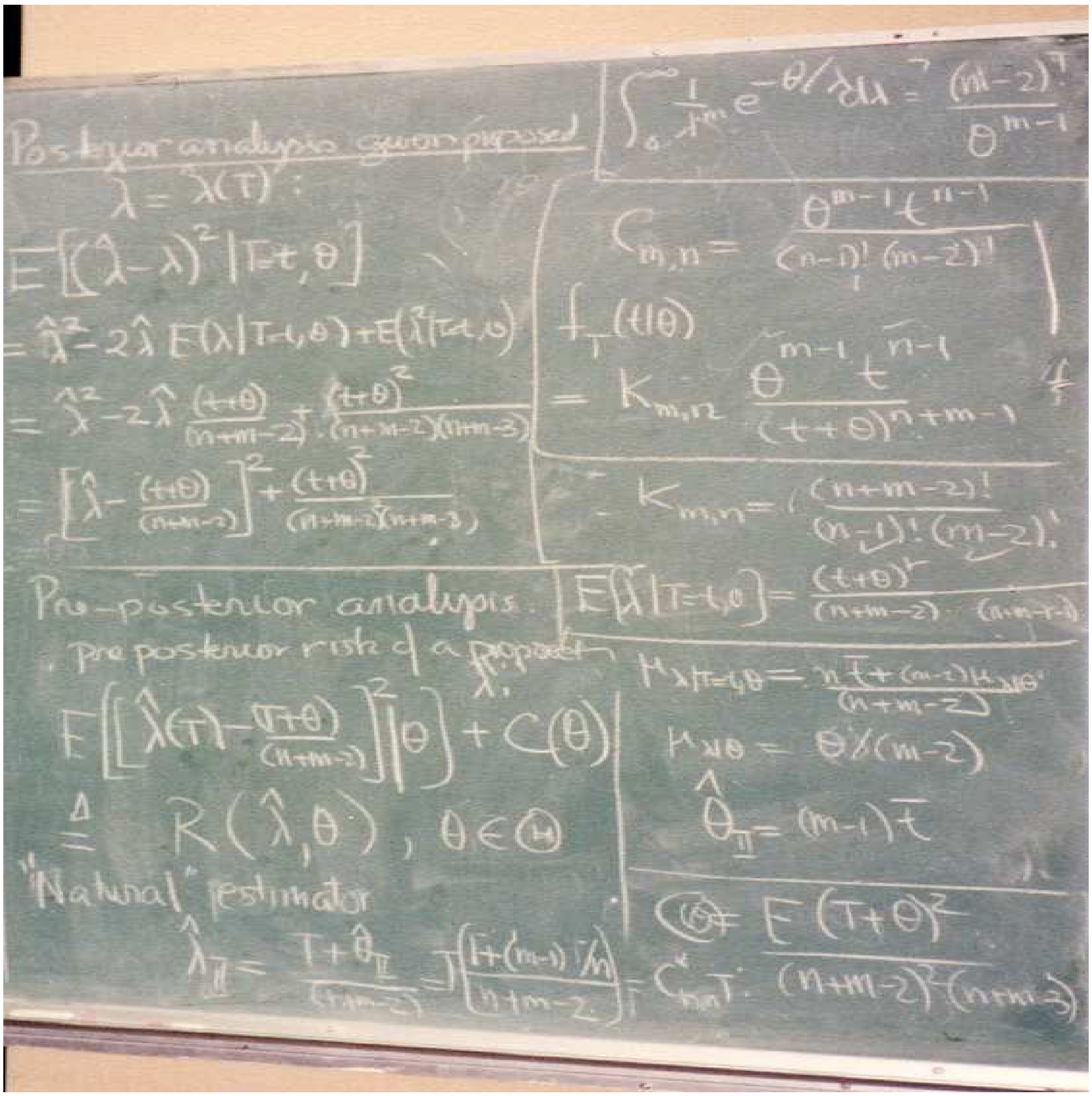
Thanks to technical assistance from
Ruben and David Zamar

I can now show you Jim's proposal
in his own handwriting

Here is the left hand side.



...and the right hand side!



Expected utility w.r.t. the marginal distribution of T

$$U_{\alpha,\beta}(\widehat{\lambda}) = \gamma_o - \gamma \mathcal{E}\left(\widehat{\lambda}(T) - \frac{\beta+T}{\alpha+n-2}\right)^2 - C(\alpha,\beta)$$

DM$_{\alpha,\beta}$'s preferred estimator is $\frac{\beta+T}{\alpha+n-2}$

Reference utility level

(Weerahandi and Zidek (1983))

We used Savage's (1954)

$\max_{\widehat{\lambda}} U_{\alpha,\beta}(\widehat{\lambda}) = \gamma_o - C(\alpha,\beta)$

Thus, we use

$$-U_{\alpha,\beta}(\hat{\lambda}) = \frac{\gamma}{(\alpha+n-2)^2}\mathcal{E}\left(\frac{\hat{\beta}(T)}{\beta} - 1\right)^2$$

where $\hat{\beta}(T) = \hat{\lambda}(T)(\alpha + n - 2) - T$

How to estimate β based on T with loss

$$\left(\frac{d}{\beta} - 1\right)^2$$

when T has the F-density

$$\propto \frac{(t/\beta)^{n-1}}{(1+t/\beta)^{n+\alpha-1}} \qquad t > 0?$$

Need a parameter space

1) $\beta > 0$;

2) $a \leq \beta \leq b$ for $0 < a < b < \infty$, a, b known.

3) $\beta \geq a$ for $a > 0$ known.

CASE 1, $\beta > 0$

$\widehat{\beta}_{MLE} = \frac{\alpha - 2}{n} T$ is inadmmisible

dominated by $\widehat{\beta}_{BE} = \frac{\alpha - 3}{n+1} T$

which is admissible and minimax

Correspondingly

$\widehat{\lambda}_{MLE} = \frac{1}{n} T$ is G-inadmissible

and dominated by $\widehat{\lambda}_{BE} = \frac{1}{n+1} T$

which is G-admissible and G-minimax

CASE 2, $a \leq \beta \leq b$, $0 < a < b < \infty$

$$\beta_{MLE} = \tfrac{T}{n} I(a \leq \tfrac{T}{n} \leq b) + a I(\tfrac{T}{n} < a) + b I(\tfrac{T}{n} > b)$$

β_{MLE} inadmissible

van Eeden and Zidek (1994b)

by using Charras (1979)

Charras and van Eeden (1991)

or, equivalently,

$$\lambda_{MLE}(T) = \tfrac{\beta_{MLE} + T}{\alpha + n - 2}$$

is group-inadmissible for estimating λ.

Dominators for β_{MLE}?

van Eeden and Zidek (1994b)

Using Charras (1979)

Charras and van Eeden (1991)

T with density $\frac{1}{\beta} f\left(\frac{t}{\beta}\right)$, $t > 0$, $a \leq \beta \leq b$

Conditions on f

$\delta(T)$ estimator of β

$P_\beta(\delta(T) = a) > 0$ and $P_\beta(\delta(T) = b) > 0$

for all $\beta \in [a, b]$

Two dominators.

First dominator:

There exists (a', b') with $a < a' < b' < b$

such that the MLE of β for $\beta \in [a', b']$

dominates β_{MLE} on $[a, b]$.

Second dominator:

There exist ε_1, ε_2 with

$a < a + \varepsilon_1 < b - \varepsilon_2 < b$ such that

$$\delta'(T) = \begin{cases} a + \varepsilon_1 & \text{if } \beta_{MLE} = a \\ b - \varepsilon_2 & \text{if } \beta_{MLE} = b \\ \beta_{MLE} & \text{if } a < \beta_{MLE} < b \end{cases}$$

dominates β_{MLE} on $[a, b]$.

Minimaxity when $a \le \beta \le b$

From a general result by

van Eeden and Zidek (1999)

When $(b/a) - 1$ is "small enough", there

exists a prior on $\{a, b\}$

for which the Bayes estimator

is unique minimax

and thus admissible.

CASE 3, $\beta \geq a$, $a > 0$ known

$\widehat{\beta}_\rho = \max(\rho T, a) \quad \rho > 0$ known

is inadmissible by Charras (1979)

and Charras and van Eeden (1991)

$\widehat{\beta}_{1/n}$ is the MLE of β

By van Eeden and Zidek (1994a and b)

$\widehat{\beta}_{(\alpha-3)/(n+1)}$ is minimax

When $\frac{\alpha-3}{n+1} < \frac{1}{n}$ or $\alpha < 4 + \frac{1}{n}$

$\widehat{\beta}_{(\alpha-3)/(n+1)}$ dominates $\widehat{\beta}_{1/n} = \beta_{MLE}$

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